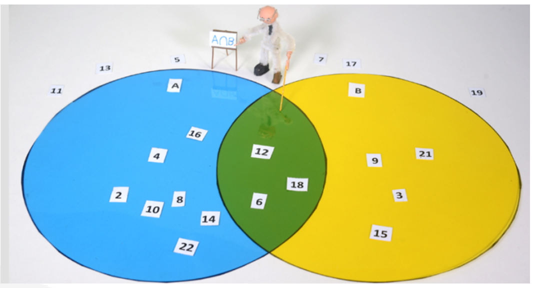
Sets and applications



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1. Sets

In mathematics, we often en counter "sets", for example, real numbers from a set. Defining a set formally is a delicate matter, so we will use "naive" set theory, based on the intuitive properties of sets.

Definition

A set* is a collection of objects called elements. We use uppercase letters to label sets, and elements will usually be represented by lower case letters. When a is an element of a set A, we write $a \in A$, otherwise, we write $a \notin A$ if A contains no elements, it is the empty set, denoted $\varnothing$ or $\{\}$. Two sets are equal if they have exactly the same elements. In other words $A = B \Leftrightarrow (x \in A \Leftrightarrow x \in B)$.

Example

The sets $\{1, 2, 3\}$ and $\{3, 2, 1\}$ are the same, because the ordering does not matter. The set $\{1, 1, 2, 3, 3\}$ is also the same set as $\{1, 2, 3\}$, because we are not interested in repetitions.

Example

The set $\{x : x \text{ is a primer number }\}$ is implicit.

Definition

The cardinal $|A|$ of a set is number of distinct elements of A. If $|A|$ is finite, the A is said to be finite. Otherwise, A is said to be infinite.

Example

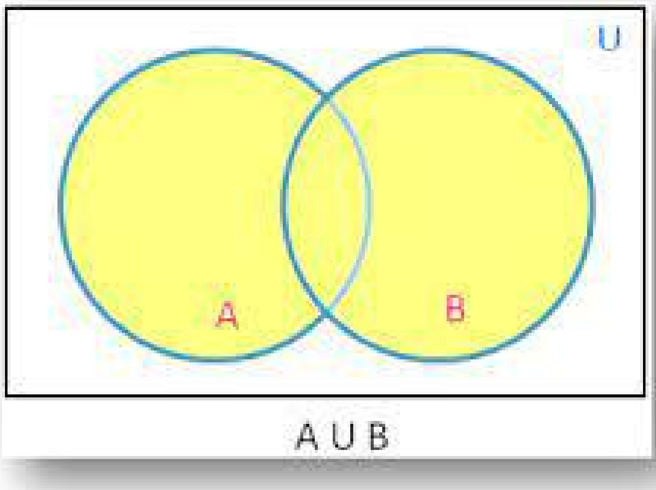
1. $|\varnothing| = 0$ while $|\{\varnothing\}| = 1$.
2. $|\{1, 2, 5\}| = 3$.
3. The set of primer numbers is infinite.

2. Set operations

We now use connectives to define the set operations*, these allow is to build new set from given ones. Let A and B be subsets of the set E.

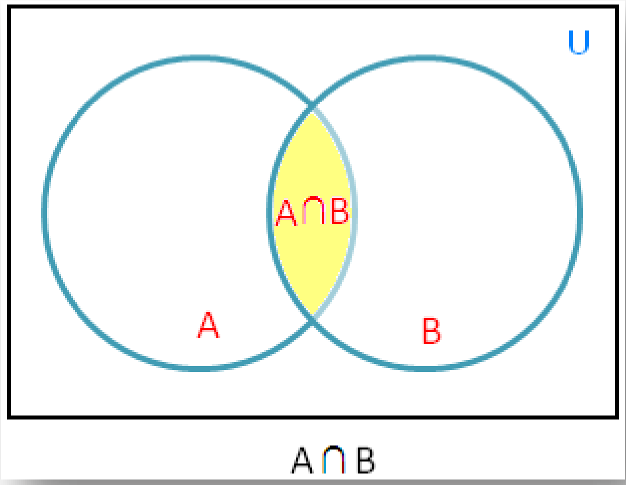
Definition: The union

The union of A and B is $A \cup B = \{x : x \in A \vee x \in B\}$ is the set of all elements that are in A or in B.



Definition: The intersection

The intersection of A and B is $A \cap B = \{x : x \in A \wedge x \in B\}$ is the set of all elements that are in both A and B.



Example

If $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$ then $A \cup B = \{1, 2, 3, 4, 5, 6\}$ and $A \cap B = \{3, 4\}$

Method

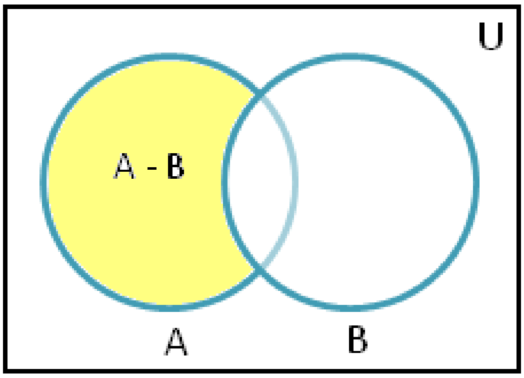
Let A and B be finite sets, then we have $|A \cup B| = |A| + |B| - |A \cap B|$.

Definition

The sets A and B disjoint when $A \cap B = \varnothing$.

Definition: The set difference

The set difference of A and B, or relative complement of B with respect to A, written $A \setminus B$ and read "A minus B" or "the complement of B with respect to A," is the set of all elements in A that are not in B. $A \setminus B = \{x : x \in A \wedge x \notin B\}$.



Difference of two sets
A - B

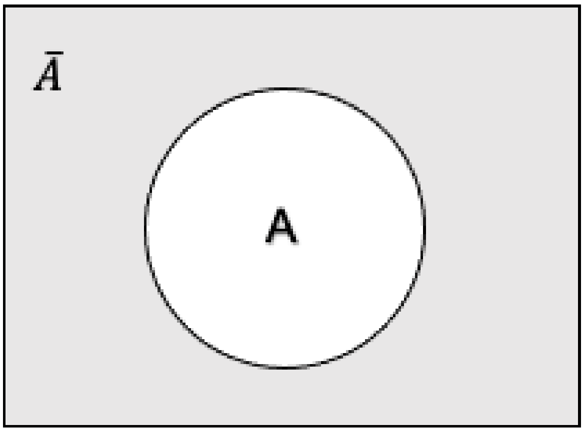
See ""

Example

1. $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,$A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5, 6, 7, 8\}$ then $A \setminus B = \{1, 2\}$, $\overline{A} = \{6, 7, 8, 9, 10\}$
2. $\mathbb{R} \setminus \mathbb{Q}$: irrational numbers.

Definition: The complement

The complement of the set A, written $C_E A$ and read "the complement of A," is the set of all elements of E that are not in A. That is, $\overline{A} = C_E A = \{x : x \in E \wedge x \notin A\}$.



Some properties

From the proprieties of the logical operations we drive the following,

Method

1. $A \cup (B \cup C) = (A \cup B) \cup C$
2. $A \cap (B \cap C) = (A \cap B) \cap C$
3. $A \cap B = B \cap A$
4. $A \cup B = B \cup A$
5. $\overline{\overline{A}} = A$
6. $\overline{A \cap B} = \overline{A} \cup \overline{B}$
7. $\overline{A \cup B} = \overline{A} \cap \overline{B}$
8. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
9. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Cartesian products

The cartesian product of two or more sets is the set of all ordered pairs/n-tuples of the sets. It is most commonly implemented in set theory. Cartesian product is the product of any two sets, but this product is actually ordered i.e, the resultant set contains all possible and ordered pairs such that the first element of the pair belongs to the first set and the second element belongs to the second set. Since their order of appearance is important, we call them first and second elements, respectively.

Definition

The Cartesian product of A and B is $A \times B = \{(a, b), a \in A \wedge b \in B\}$

Example

1. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is called the Cartesian plane.
2. If $A = \{1, 2\}$ and $B = \{0, 1, 2\}$ then $A \times B = \{(1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\}$.

✎ Note

$A \times B \times C \times D = \{(a, b, c, d), a \in A, b \in B, c \in C, d \in D\}$ a n d $A^n = \{(a_1, a_2, \ldots, a_n) / a_i \in A, i = 1, \ldots, n\}$.

Sub sets

Subsets are a core concept in the study of Set Theory*, similar to Sets.

Definition

A set A is a subset of another set B if all elements of the set A are elements of the set B. We write $A \subset B \Leftrightarrow \forall x, x \in A \Rightarrow x \in B$. If A is not a subset of B, we write $A \subsetneq B$.

Example

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Method

Let A, B, C be sets, then

1. $\varnothing \subset A$.
2. $A = B \Leftrightarrow A \subset B \wedge B \subset A$.
3. If $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$.

Families of sets

The elements of a set may themselves be sets. For example, the power set of a set A , $P(A)$ is the set of all subsets of A . The phrase, “a set of sets” sounds confusing, and so we often use the terms collection and family when we wish to emphasize that the elements of a given set are themselves sets. We would then say that the power set of A is the family (or collection) of sets that are subsets of A .

Definition

For any set A, $P(A) = \{B, B \subseteq A\}$.

Example

The set $A = \{1, 2, 3\}$ then $P(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

3. Applications

Let E, F be sets. A function $f : E \to F$ assigns to each $x \in E$ a unique element $f(x) \in F$. Functions are also called maps. mappings or application.

Definition

Let $f : E \to F$ be a function. Then E is called the domain of f and F is called the codomain of f. We write $f : x \to f(x)$ to indicate that is the function that maps x to $f(x)$.

Example

Let $E = \{1, 2, 3\}$ and $F = \{a, b\}$. Then we can define a function $f : E \to F$ by setting $f(1) = f(2) = a$ and $f(3) = b$. a is the image of 1 inder f, 1 is the perimage of a under f.

Injective function

The function is injective, if each element of the codomain is mapped to by at most one element of the domain, or equivalently, if distinct elements of the domain map to distinct elements in the codomain.

Definition

A function $f : E \to F$ is injective if we have: $\forall (x_1, x_2) \in E^2 : f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. An injection is also know as a one to one function.

Example

- The function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2, \forall x \in \mathbb{Z}$ is not injective since $f(1) = f(-1)$
- The function $g : \mathbb{N} \to \mathbb{N}$ defined by $g(x) = x^2$ is injective.

Surjective function

The function is surjective, if each element of the codomain is mapped to by at least one element of the domain. That is, the image and the codomain of the function are equal.

Definition

A function $f : E \to F$ is surjective if we have $\forall y \in F, \exists x \in E, y = f(x)$.

Example

The function $\mathbf{R} \to [-1, 1] : x \mapsto \sin(x)$ is surjective.

To watch the video click *here*.

4. Exercice : Set

[solution n°1 p.10]

Let $A = \left\{x \in \mathbb{R} : (x+5)^2 = 9^2\right\}$. In what form can we still write the set A?

- $A = \{4\}$
- $A = \varnothing$
- $A = \{-14\}$
- $A = \{4, -14\}$

5. Exercice : Application

[solution n°2 p.10]

Let $f : [-1, 1] \to [-1, 1]$ be the map defined by: $\forall x \in [-1, 1]\,,\, f(x) = \dfrac{2x}{1+x^2}$

What are the correct answers?

- [ ] f is injective.
- [ ] f is surjective.
- [ ] f is injective but not surjective.
- [ ] f is surjective but not injective.
- [ ] f is not surjective and not injective.
- [ ] f is bijective and $f^{-1}(x) = \dfrac{x}{1+\sqrt{1-x^2}}$.

Exercises solution

> **Solution** n°1

Exercice p. 9

Let $A = \left\{x \in \mathbb{R} : (x+5)^2 = 9^2\right\}$. In what form can we still write the set A?

- ○ $A = \{4\}$
- ○ $A = \varnothing$
- ○ $A = \{-14\}$
- ⊙ $A = \{4, -14\}$

> **Solution** n°2

Exercice p. 9

Let $f : [-1, 1] \to [-1, 1]$ be the map defined by: $\forall x \in [-1, 1], f(x) = \dfrac{2x}{1+x^2}$

What are the correct answers?

- [x] f is injective.
- [x] f is surjective.
- [ ] f is injective but not surjective.
- [ ] f is surjective but not injective.
- [ ] f is not surjective and not injective.
- [x] f is bijective and $f^{-1}(x) = \dfrac{x}{1+\sqrt{1-x^2}}$.

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