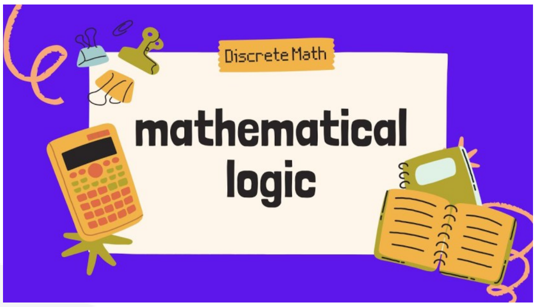
Chapter 01: Logic concepts



Dr. Hind BOUREDJI

Mohamed Khider University of Biskra

Faculty of exact sciences, natural and life sciences

Department of Mathematics

Table of contents

I Logic concepts

1. Definitions

Proposition

A proposition is a central concept in logic, often characterized as the primary bearer of truth or falsity.

Fundamental

Aproposition* is a mathematical precise statement that is either true or false, but not both.

Example

1. The statement "$1 + 1 > 3$" is false while statement "$5 > 3$" is true, both statement are propositions.
2. The statement "What a great Boole!" is not a proposition some one is simply expressing an opinion.
3. The statement "$x + 1 < 7$" is not a proposition, the truth value of this statement relies on what the variable x is assigned.
4. The statement "$5 + 7$" is not a proposition, because it has a truth value, namely true.

Paradox

A paradox is a statement that can not be assigned a truth value.

Example

"This statement is false". If true, than false and if false then true.

Example

A classic example of a paradox is the barber paradox, it states the barber shaves every man in town who does not shave him self. If the barber shaves him self, then this means the barber does not shave him self and if he does not shave himself then he must shave himself, neither is possible and so it is a paradox.

For more details see*.

2. Operators

We are particularly interested in combining propositions by operators.

Definition

A compound proposition is a statement obtained by combining propositions with logical operators.

Conjunction

The conjunction operator is the binary operator which, when applied to two propositions p and q, yields the proposition "p and q". This time, for $p \wedge q$ to be true, we need both p and q to be true, note that $(\wedge)$ is also commutative.

p	q	$p \wedge q$	$q \wedge p$
T*	T	T	T
T	F	F	F
F*	T	F	F
F	F	F	F

Truth Table of conjunction

Disjunction

The disjunction operator is the binary operator which, when applied to two propositions p and q, yields the proposition "p or q". The disjunction operator returns T when at last one of the two propositions p or q is true. The operator $\vee$ is commutative*.

p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

Truth Table of disjunction

Note

More than two propositions can be joined using logical operators. In this instances, it is important to be careful about how they are grouped.

Example

1. $[(2+1=4) \wedge (2+2=5)] \vee [5-2=3]$ is true.
2. $[(2+1=4) \wedge (2+2=5)] \vee [5+2=3]$ is false.

Negation

The negation operator is a unary operator. Sometimes in mathematics it's important to determine what the opposite of a given mathematical statement is. This is usually referred to as "negating" a statement. One thing to keep in mind is that if a statement is true, then its negation is false (and if a statement is false, then its negation is true).

p	$\overline{p}$
T	F
F	T

Truth Table of negation

Implication

In logic, implication is relationship between different propositions where the second proposition is a logical consequence of the first. The statement $\overline{p} \vee q$ is denoted by $p \Rightarrow q$. One way to think of the meaning of $p \Rightarrow q$ is to consider it a contract that says if the first condition is satisfied, then the second will also be satisfied. We say,

1. p implies q.
2. If p then q.

p	q	$\overline{p}$	$q \Rightarrow p \equiv \overline{p} \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Truth Table of implication

the converse of $p \Rightarrow q$ is $q \Rightarrow p$. The contapositive of $p \Rightarrow q$ is $\overline{q} \Rightarrow \overline{p}$.

Equivalence

Logical equivalence is the condition of equality that exists between two statements or sentences in propositional logic. The relationship between the two statements translates verbally into "if and only if." In mathematics, logical equivalence is typically symbolized by a double arrow $(\Leftrightarrow)$. The expression $p \Leftrightarrow q$ means $(p \Rightarrow q \text{ and } q \Rightarrow p)$.

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Truth Table of equivalence

See ""

Equivalent propositions

Two propositions are equivalent if they have identical truth tables.

1. $p \wedge q \Leftrightarrow q \wedge p$
2. $p \vee q \Leftrightarrow q \vee p$
3. $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$
4. $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
5. $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
6. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
7. $\overline{\overline{p}} \Leftrightarrow p$
8. $\overline{p \wedge q} \Leftrightarrow \overline{p} \vee \overline{q}$
9. $\overline{p \vee q} \Leftrightarrow \overline{p} \wedge \overline{q}$
10. $p \Rightarrow q \Leftrightarrow \overline{q} \Rightarrow \overline{p}$
11. $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow p \Rightarrow r$
12. $(p \Leftrightarrow q) \wedge (q \Leftrightarrow r) \Rightarrow p \Leftrightarrow r$

3. Predicates and quantifiers

A predicate is an expression of one or more variables determined on some specific domain. A predicate with variables can be made a proposition by either authorizing a value to the variable or by quantifying the variable.

Definition

A predicate is statement that contains variables. A predicate may be true or false depending on the values of these variables.

Universal quantifier

$\forall x \,(\text{for all } x)\,,\, P\,(x)$ means that the predicate $P\,(x)$ is true for all possible values of x.

A universal quantification is a type of quantifier*, a logical constant which is interpreted as "given any", "for all", or "for any". It expresses that a predicate can be satisfied by every member of a domain of discourse. In other words, it is the predication of a property or relation to every member of the domain. It asserts that a predicate within the scope of a universal quantifier is true of every value of a predicate variable.

Example

1. $\forall x, \left(x^2 \geq 0\right)$, i.e., "the square of any number is not negative".
2. $\forall x, (x \text{ is a square} \Rightarrow x \text{ is a rectangle})$, i.e., "all squares are rectangles."

Existential quantifier

An existential quantification is a type of quantifier, a logical constant which is interpreted as "there exists", "there is at least one", or "for some".

$\exists x \text{ (there exists } x) \; P(x)$ means that there exists an $x \; P(x)$ is true. Sometimes, we will use also $\exists x, P(x)$. It means that there exists a unique x where $P(x)$ is true.

Example

1. $\forall x \in \mathbb{Z} : \left(x^2 = 5 \Rightarrow x = 2\right)$ is true.
2. $\exists x \in \mathbb{R}^2 : x^2 = 2$ is true, but $\exists x \in \mathbb{Z} : x^2 = 2$ is false.
3. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y > x$ is true. The order of the quantifiers is very important, this statement $\exists y \in \mathbb{R} \forall x \in \mathbb{R}, y > x$ is false.

A quantified propositional function is a statement; thus, like statements, quantified functions can be negated. The negation of a propositional function's existential quantification is a universal quantification of that propositional function's negation.

$$\overline{(\forall x, P(x))} \Leftrightarrow \left(\exists x \,, \overline{P(x)}\right)$$

$$\overline{(\exists x, P(x))} \Leftrightarrow \left(\forall x \,, \overline{P(x)}\right)$$

Example

1. Statement: $\forall x \in \mathbb{R} : 2x < 3$ and negation: $\exists x \in \mathbb{R} : 2x \geq 3$
2. Statement: $\exists x \in \mathbb{R} : 2^x = 256$ and negation: $\forall x \in \mathbb{R} : 2^x \neq 256$
3. Statement: $\exists y \in \mathbb{R}, \forall x \in \mathbb{R} : y > x$ and negation: $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, y > x$

4. Methods of proof

Mathematical statements can typically be phrased as an implication, may be complex statements themselves that involve conjunctions (and), disjunctions (or), negations, quantifiers, even implications. There are various ways in which an implication can be proven true, and there is no hard and fast rule that dictates which proof method to use given a particular problem.

Direct proof

A direct proof is one of the most familiar forms of proof. We use it to prove statements of the form ”if p then q” or ”p implies q” which we can write as $p \Rightarrow q$. The method of the proof is to takes an original statement p, which we assume to be true, and use it to show directly that another

statement q is true. So a direct proof has the following steps:

- Assume the statement p is true.
- Use what we know about p and other facts as necessary to deduce that another statement q is true, that is show $p \Rightarrow q$ is true.

Example

Let $(a, b) \in \mathbb{Z}^2$ show that, $a + b\sqrt{2} = 0 \Rightarrow a = b = 0$.

Proof by contrapositive

Proof by contrapositive, or proof by contraposition, is a rule of inference used in proofs, where one infers a conditional statement from its contrapositive. In other words, the conclusion "if p, then q" is inferred by constructing a proof of the claim "if not q, then not p" instead. More often than not, this approach is preferred if the contrapositive is easier to prove than the original conditional statement itself. We show $\overline{q} \Rightarrow \overline{p}$ insted of $p \Rightarrow q$.

Example

---

let $n \in \mathbb{N}^*$ show that $n^2 - 1$ is not divisible by $8 \Rightarrow n$ is even.

Proof by contraduction

Proof by contradiction (also known as indirect proof) is a common proof technique that is based on a very simple principle: something that leads to a contradiction can not be true, and if so, the opposite must be true. To show that p is true, we suppose that p is false and than $\overline{p}$ is true. We show that $\overline{p} \Rightarrow \overline{q}$, when q is true. So $q \Rightarrow p$ is true. As q is true then p is true.

Example

Show that 0 does not admit an inverse in $\mathbb{R}$.

By induction

A proof by induction is just like an ordinary proof in which every step must be justified. Let $P(x)$ be a logical statement for each $n \in \mathbb{N}$. The principe of mathematical induction states that $P(n)$ is true all $n \in \mathbb{N}$ if $P(0)$ is true, and $P(n) \Rightarrow P(n+1)$ for all $n \in \mathbb{N}$.

Example

Show that for all $n \in \mathbb{N}$, we have $2^{n-1} \leq n! \leq n^n$.

By giving a counter example

A proof by counter example is not technically a proof. It is merely a way of showing that a given statement cannot possibly be correct by showing an instance that contradicts a universal statement. To show that a proposition p is false, we must show that $\overline{p}$ is true.

Example

Show that the proposition:

1. $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, f(x) = y$ is false.
2. $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2, \forall (x, y) \in \mathbb{R}^2 : f(x) = f(y) \Rightarrow x = y$ is false.

To watch the video click *here*.

5. Exercice : The logical proposition

[solution n°1 p.10*]*

Is the logical proposition $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : 2x + y > 0$ true or false?

- True
- False

6. Exercice : The logical proposition

[solution n°2 p.10*]*

Among the following statements, identify the propositions

- [ ] $4 + 2$
- [ ] $4 + 2 = 5$
- [ ] $\forall x \in \mathbb{N}, x < 5$
- [ ] $x + 2 = 2x$
- [ ] Can you help me?
- [ ] 5 is prime number.

Exercises solution

> **Solution** n°1

Exercice p. 9

Is the logical proposition $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : 2x + y > 0$ true or false?

- (●) True
- ( ) False

> **Solution** n°2

Exercice p. 9

Among the following statements, identify the propositions

- [ ] $4 + 2$
- [x] $4 + 2 = 5$
- [ ] $\forall x \in \mathbb{N}, x < 5$
- [ ] $x + 2 = 2x$
- [ ] Can you help me?
- [x] 5 is prime number.

Glossary

Commutative

property of an operation which allows you to change the order of the terms without changing the result.

quantifier

A quantifier is an operator that specifies how many individuals in the domain of discourse satisfy an open formula

Abbreviation

F: False

T: True

Bibliography

O'LEARY, Michael L. A first course in mathematical logic and set theory. John Wiley & Sons, 2015.

C. Degrave et D. Degrave, Algèbre 1ère année : cours, méthodes, exercices résolus, Bréal, 2003.