University of Biskra
Mathematics Department
Module: Analysis 2

First year license
2024/2025

Final Exam

Exercise 1 ***(6pts=3pts+3pts)*** Compute the following integrals:

$$\int \frac{x^2+1}{x(x-1)(x+1)}dx \quad \text{and} \quad \int \ln(x^2+x+1)dx.$$

Exercise 2 ***(4pts=1pt+2pts+1pt)*** Consider the real function f on D_f defined by:

$$f(x) = \sqrt{a^2-x^2}, \quad \textit{with } a > 0.$$

1. Give the domain D_f.
2. Calculate the value of the area delimited by the curve of f and the axis $y = 0$ over the domain D_f.
3. Deduce the expression for the surface delimited by a circle with center $(0, 0)$ and radius R.

We give: $\sin(1) = -\sin(-1) = \frac{\pi}{2}$.

Exercise 3 ***(6pts=4pts+2pts)*** Let us consider the following linear differential equation:

$$y' - \frac{y}{(x+1)^n} = \frac{1}{(x+1)^n}, \quad \text{with } x,\ y \in \mathbb{R}_+ \text{ and } n \in \mathbb{R}. \tag{1}$$

1. Solve the equation (2) according to the values of n.
2. Determine the solution of (2) that satisfies the condition $y(0) = 0$ when $n = 1$ and $n = 2$.

Exercise 4 ***(4pts)*** Find the global solution of the following differential equation

$$y'' + 2y' - 3y = 3x + 4 - 4e^{-3x} + 5\sin(2x+1).$$

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Final Exam Solution

Solution of the Exercise 1

1. To compute the antiderivative we use the decomposition approach.

$$\int \frac{x^2+1}{x(x-1)(x+1)}dx = \int \frac{a}{x} + \frac{b}{x-1} + \frac{c}{x+1}dx$$

We have

- $\frac{x^2+1}{(x-1)(x+1)} = a + \frac{bx}{x-1} + \frac{cx}{x+1}$ for $x = 0$ we get $a = -1$.
- $\frac{x^2+1}{x(x+1)} = \frac{a(x-1)}{x} + b + \frac{c(x-1)}{x+1}$ for $x = 1$ we get $b = 1$.
- $\frac{x^2+1}{x(x-1)(x+1)} = \frac{a(x+1)}{x} + \frac{b(x+1)}{x-1} + c$ for $x = -1$ we get $c = 1$.

$$\begin{aligned}\int \frac{x^2+1}{x(x-1)(x+1)}dx &= \int \frac{-1}{x} + \frac{1}{x-1} + \frac{1}{x+1}dx \\ &= -\ln(|x|) + \ln(|(x-1)|) + \ln(|(x+1)|) + c. \\ &= \ln\left(\left|\frac{x^2-1}{x}\right|\right) + c.\end{aligned}$$

2. To compute the considered integral we use the integration by parts.

$$\int \ln(x^2+x+1)dx = \int 1 \times \ln(x^2+x+1)dx.$$

we take

$$\begin{cases} u' &= 1 \\ v &= \ln(x^2+x+1) \end{cases} \Longrightarrow \begin{cases} u &= x \\ v' &= \frac{2x+1}{x^2+x+1} \end{cases}$$

$$\begin{aligned}\int \ln(x^2+x+1)dx &= x\ln\left(x^2+x+1\right) - \int \frac{2x^2+x}{x^2+x+1}dx. \\ &= x\ln\left(x^2+x+1\right) - \int 2 - \frac{x+2}{x^2+x+1}dx. \\ &= x\ln\left(x^2+x+1\right) - \int 2dx - \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx - \frac{3}{2}\int \frac{1}{x^2+x+1}dx. \\ &= x\ln\left(x^2+x+1\right) - \int 2dx - \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx - \frac{3}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}dx. \\ &= x\ln\left(x^2+x+1\right) - 2x + \ln(x^2+x+1) - \frac{3}{2}\frac{2}{\sqrt{3}}\arctan\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c.\end{aligned}$$

Solution of the Exercise 2

1. The domain D_f is defined as follows:

$$D_f = \{x \in \mathbb{R} : a^2 - x^2 \geq 0\} = [-a; a].$$

2. The area delimited by the curve of f and the axis $y = 0$ over the entire domain D_f is:

$$S = \int_{D_f} |f| dx = \int_{-a}^{a} \sqrt{a^2 - x^2} dx = \int_{-a}^{a} \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} dx.$$

We have

$$\begin{aligned}
\int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} dx &= (\alpha x + \beta) \times \left(\sqrt{a^2 - x^2}\right) + \lambda \int \frac{1}{\sqrt{a^2 - x^2}} dx \\
\Longrightarrow & \left[\int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} dx\right]' = \left[(\alpha x + \beta)\sqrt{a^2 - x^2} + \int \frac{\lambda}{\sqrt{a^2 - x^2}}\ dx\right]' \\
\Longrightarrow & \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} = \frac{-2\alpha x^2 - \beta x + \lambda + \alpha^2 a^2}{\sqrt{a^2 - x^2}} \\
\Longrightarrow & \left\{\begin{array}{lcl} \alpha & = & \frac{1}{2} \\ \beta & = & 0 \\ \lambda & = & \frac{a^2}{2} \end{array}\right. ,
\end{aligned}$$

thus,

$$\begin{aligned}
\int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} dx &= \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \int \frac{1}{\sqrt{a^2 - x^2}} dx\right) \\
&= \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \arcsin\left(\frac{x}{a}\right)\right) + c
\end{aligned}$$

Consequently,

$$S = \int_{D_f} \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} dx = \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \arcsin\left(\frac{x}{a}\right)\right) + c\Big|_{-a}^{a} = a^2 \pi.$$

It's to be noted that the integral in question can be also computed using the integration by parts. Indeed, we have:

$$\begin{aligned}
I(x) &= \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} dx = \int \frac{a^2}{\sqrt{a^2 - x^2}} + \int \frac{-x^2}{\sqrt{a^2 - x^2}} \\
&= a^2 \arcsin\left(\frac{x}{a}\right) + \int x \times \frac{-2x}{2\sqrt{a^2 - x^2}} dx.
\end{aligned}$$

To compute the last integral we take

$$\left\{\begin{array}{lcl} u & = & x \\ v' & = & \frac{-2x}{2\sqrt{a^2 - x^2}} \end{array}\right. \Longrightarrow \left\{\begin{array}{lcl} u' & = & 1 \\ v & = & \sqrt{a^2 - x^2} \end{array}\right.$$

So,

$$\begin{aligned}
I(x) &= a^2 \arcsin\left(\frac{x}{a}\right) + x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2} \\
&= a^2 \arcsin\left(\frac{x}{a}\right) + x\sqrt{a^2 - x^2} - I(x) \\
&= \frac{a^2}{2} \arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2}
\end{aligned}$$

Consequently,

$$S = I(a) - I(-a) = a^2\pi.$$

3. Recall that the equation of a circle with center $(0,0)$ and radius R is given by

$$\begin{aligned} x^2+y^2=R^2 \quad &\Longrightarrow \quad y=\pm\sqrt{R^2-x^2} \\ &\Longrightarrow \quad S=\int_{-R}^{R}|\sqrt{a^2-x^2}|dx+\int_{-R}^{R}|-\sqrt{a^2-x^2}|dx=2\int_{-R}^{R}\sqrt{a^2-x^2}dx=2R^2\pi. \end{aligned}$$

Solution of the Exercise 3

$$y' - \frac{y}{(x+1)^n} = \frac{1}{(x+1)^n}, \quad \text{with } x,\ y \in \mathbb{R}_+ \text{ and } n \in \mathbb{R}. \tag{2}$$

At first glance, the equation (2) is a linear differential equation with a right-hand side, but a little articulation can transform it into a linear differential equation without a right-hand side.

$$\begin{aligned} y' - \frac{y}{(x+1)^n} = \frac{1}{(x+1)^n} \quad &\Longrightarrow \quad y' - \frac{y}{(x+1)^n} - \frac{1}{(x+1)^n} = 0 \\ &\Longrightarrow \quad y' - \frac{y+1}{(x+1)^n} = 0 \\ &\Longrightarrow \quad \frac{y'}{y+1} = \frac{1}{(x+1)^n} \\ &\Longrightarrow \quad \int \frac{1}{y+1}dy = \int \frac{1}{(x+1)^n}dx. \end{aligned}$$

At this level, we distinguish two possible cases, namely the case $n = 1$ and the case $n \neq 1$.

Case $n = 1$ In the case we get

$$\begin{aligned} \int \frac{1}{y+1}dy = \int \frac{1}{(x+1)}dx \quad &\Longrightarrow \quad \ln(y+1) = \ln(x+1) + c \\ &\Longrightarrow \quad y+1 = K\ (x+1); \quad \text{with } K \in \mathbb{R}_+. \\ &\Longrightarrow \quad y = K\ (x+1) - 1; \quad \text{with } K \in \mathbb{R}_+. \end{aligned}$$

Case $n \neq 1$ In this case we have

$$\begin{aligned} \int \frac{1}{y+1}dy = \int \frac{1}{(x+1)^n}dx \quad &\Longrightarrow \quad \ln(y+1) = \frac{(x+1)^{1-n}}{1-n} + c \\ &\Longrightarrow \quad y+1 = K\ e^{f(x)}; \quad \text{with } f(x) = \frac{(x+1)^{1-n}}{1-n} \text{ and } K \in \mathbb{R}_+. \\ &\Longrightarrow \quad y = K\ e^{f(x)} - 1; \quad \text{with } f(x) = \frac{(x+1)^{1-n}}{1-n} \text{ and } K \in \mathbb{R}_+. \end{aligned}$$

While if we consider that the differential equation (2) is a linear differential equation with second membre, so to solve it we must find its homogenous solution then its particular solution.

1. Solve the equation (2) according to the values of n.

   (a) Let's find the homogenous solution of the given equation.

$$y' - \frac{y}{(x+1)^n} = 0 \Longrightarrow \frac{y'}{y} = \frac{1}{(x+1)^n} \Longrightarrow \int \frac{1}{y}dy = \int \frac{1}{(x+1)^n}dx. \tag{3}$$

   At this level we distinguish two possible cases, namely the Case $n = 1$: and the case $n \neq 1$.

Case $n = 1$**:** In the case (3) becomes

$$\int \frac{1}{y} dy = \int \frac{1}{(x+1)} dx \implies \ln(y) = \ln(x+1) + c$$
$$\implies y_h = K\ (x+1); \quad \text{with } K \in \mathbb{R}_+.$$

Case $n \neq 1$**:** In this case we have

$$\int \frac{1}{y} dy = \int \frac{1}{(x+1)^n} dx \implies \ln(y) = \frac{(x+1)^{1-n}}{1-n} + c$$
$$\implies y_h = K\ e^{f(x)}; \quad \text{with } f(x) = \frac{(x+1)^{1-n}}{1-n} \text{ and } K \in \mathbb{R}_+.$$

(b) Now let's find the particular solution of the equation by the variation of the constant.

Case $n = 1$**:** From the homogenous solution we deduce that the particular solution has the following form $y_p = K(x)(x+1)$ this $y_p^{'} = K'(x)(x+1) + K(x)$. So,

$$K'(x)(x+1) + K(x) - \frac{K(x)(x+1)}{(x+1)} = \frac{1}{x+1} \implies K'(x) = \frac{1}{(1+x)^2}$$
$$\implies K(x) = \frac{-1}{x+1} + c \text{ with } c \in \mathbb{R}.$$
$$\implies y_g = y_h + y_p = c(x+1) - 1; \quad \text{with } c \in \mathbb{R}.$$

Case $n \neq 1$**:** From the homogenous solution we deduce that the particular solution has the following form $y_p = K(x)e^{f(x)}$ then $y_p^{'} = K'(x)e^{f(x)} + K(x)f'(x)e^{f(x)}$, with $f(x) = \frac{(x+1)^{1-n}}{1-n}$ and $f'(x) = \frac{1}{(x+1)^n}$. So,

$$K'(x)e^{f(x)} + K(x)f'(x)e^{f(x)} - K(x)e^{f(x)}f'(x) = f'(x) \implies K'(x)e^{f(x)} = f'(x)$$
$$\implies K'(x) = f'(x)e^{-f(x)}$$
$$\implies K(x) = -e^{-f(x)} + c$$
$$\implies y_g = ce^{f(x)} - 1.$$

2. Determine the solution of (2) that satisfies the condition $y(0) = 0$ when $n = 1$.

Case $n = 1$**:**

$$y(0) = 0 \implies c - 1 = 0 \implies c = 1 \implies y_g = x.$$

Case $n = 2$ **:** When $n = 2$, we get $y_g = ce^{\frac{-1}{x+1}} - 1$ and as $y(0) = 0$ then $c = e^1 \implies y_g = e^{\frac{x}{x+1}} - 1$.

Solution of the Exercise 4

Let's solve the following equation:

$$y'' + 2y' - 3y = 3x + 4 - 4e^{2x} + 5\sin(2x+1). \tag{4}$$

Homogenous solution: The homogenous solution is the solution of the following equation:

$$y'' + 2y' - 3y = 0 \implies R^2 + 2R - 3 = 0 \implies \Delta = 16 > 0 \implies R_1 = 1 \text{ and } R_2 = -3.$$

As the equation admits a double solution then the homogenous solution is given by follow:

$$y_h = C_1 e^x + C_2 e^{-3x}.$$

Particulars solutions From the given equation we can distinguish three particular solutions.

1. The first particular solution corresponding to $3x+4$ have the form $y_{p_1} = ax+b$ so,$y'_{p_1} = a$ and $y''_{p_1} = 0$. By substitution of y_{p_1}, y'_{p_1} and y''_{p_1} in (4) we get

$$0 + 2a - 3(ax+b) = 3x+4 \quad \Longrightarrow \left\{ \begin{array}{lcl} -3a & = & 3 \\ 2a - 3b & = & 4 \end{array} \right. \Longrightarrow \left\{ \begin{array}{lcl} a & = & -1 \\ b & = & -2 \end{array} \right.$$

$$\Longrightarrow \quad y_{p_1} = -x - 2.$$

2. The second particular solution corresponding to $-4e^{-3x}$ have the form $y_{p_2} = axe^{-3x}$ so,$y'_{p_2} = (-3ax+a)e^{-3x}$ and $y''_{p_2} = (9ax - 6a)e^{-3x}$. By substitution of y_{p_2}, y'_{p_2} and y''_{p_2} in (4) we get

$$\left((9ax-6a)e^{-3x}\right) + 2\left((-3ax+a)e^{-3x}\right) - 3\left(axe^{-3x}\right) = -4e^{-3x} \quad \Longrightarrow \quad -4a = -4 \Longrightarrow \ a = 1$$

$$\Longrightarrow \quad y_{p_2} = xe^{-3x}.$$

3. The third particular solution corresponding to $\sin(2x+1)$ have the form $y_{p_3} = a\sin(2x+1) + b\cos(2x+1)$ so, $y'_{p_2} = -2b\sin(2x+1) + 2a\cos(2x+1)$ and $y''_{p_2} = -4a\sin(2x+1) - 4b\cos(2x+1)$.
By substitution of y_{p_1}, y'_{p_1} and y''_{p_1} in (4) we get

$$\begin{aligned} & (-7a - 4b)\sin(2x+1) + (4a - 7b)\cos(2x+1) = 5\sin(2x+1) \\ \Longrightarrow & \left\{ \begin{array}{lcl} 7a + 4b & = & -5 \\ 4a - 7b & = & 0 \end{array} \right. \\ \Longrightarrow & \left\{ \begin{array}{lcl} a & = & \frac{-7}{13} \\ b & = & \frac{-4}{13} \end{array} \right. \\ \Longrightarrow & \ y_{p_4} = -\frac{7}{13}\sin(2x+1) - \frac{4}{13}\cos(2x+1) \end{aligned}$$

We conclude that the global solution of the equation is

$$\begin{aligned} y_g &= y_h + y_{p_1} + y_{p_2} + y_{p_3} \\ &= C_1 e^x + (C_2 + x)e^{-3x} - x - 2 - \frac{7}{13}\sin(2x+1) - \frac{4}{13}\cos(2x+1). \end{aligned}$$