University of Biskra
Mathematics Department
Module: Analysis 2

First year license
2024/2025

Continuous assessment
Test N° 1

Exercise: (./10pts) Calculate the following integrals:

1. $$\int \frac{e^{-x}}{\sqrt{1-e^{-2x}}}dx,$$

   Indication: use the substitution of the variable.

2. $$\int (x+2)\cos(2x)dx,$$

   Indication: use the integration by parts.

3. $$\int \frac{x^2+x+2}{x(x^2+1)}dx,$$

   Indication: use the decomposition of the fraction.

4. $$\int \ln\left(\frac{1}{\sqrt{x^2+1}}\right)dx,$$

   Indication: use the integration by parts and the decomposition of the fraction.

Good luck

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Solution of the continuous assessment Test N° 1

1. To calculate the first integral, we use the following change of variable:

$$t = e^{-x} \Longrightarrow dt = -e^{-x}dx.$$

So we have:

$$\begin{aligned}\int \frac{e^{-x}}{\sqrt{1-e^{-2x}}}dx &= \int \frac{-1}{\sqrt{1-t^2}}dt = \arccos(t) + c = \\ &= \arccos\left(e^{-x}\right) + c = -\arcsin\left(e^{-x}\right) + c \qquad \text{with } c \in \mathbb{R}.\end{aligned}$$

2. To calculate the given integral, we will perform an integration by parts. To do this, we take the following elements:

$$\left\{\begin{array}{lcl} u &=& x+2 \\ v' &=& \cos(2x)\end{array}\right. \Longrightarrow \left\{\begin{array}{lcl} u' &=& 1 \\ v &=& \frac{1}{2}\sin(2x)\end{array}\right.$$

Thus,

$$\begin{aligned}\int (x+2)\cos(2x)dx &= \frac{1}{2}(x+2)\sin(2x) - \int \frac{1}{2}\sin(2x)dx \\ &= \frac{1}{2}(x+2)\sin(2x) + \frac{1}{4}\cos(2x) + c \qquad \text{with } c \in \mathbb{R}.\end{aligned}$$

3. The given integral is a rational function; to calculate its antiderivative, it must first be decomposed into a simple rational sum. Note that the function in question can be decomposed as follows:

$$\frac{x^2+x+2}{x(1+x^2)} = \frac{a}{x} + \frac{bx+c}{1+x^2}. \tag{1}$$

Now let's determine the value of the parameters a, b and c.

(a) Multiplying both sides of the equality (1) by x, we obtain

$$\frac{x^2+x+2}{(1+x^2)} = a + \frac{(bx+c)x}{1+x^2}.$$

Setting $x = 0$ in this last equality, we conclude that:

$$a = 2.$$

(b) To determine the values of c and d, let us calculate both sides of the equality (1) for $x = 1$ and $x = -1$, respectively. In this case, we obtain the following system of linear equations:

$$\left\{\begin{array}{lcl} 2 &=& a + \frac{b}{2} + \frac{c}{2} \\ -1 &=& -a - \frac{b}{2} + \frac{c}{2}\end{array}\right. \text{ as } a = 2 \Longrightarrow \left\{\begin{array}{rcl} b+c &=& 0 \\ -b+c &=& 2\end{array}\right. \Longrightarrow \left\{\begin{array}{lcr} b &=& -1 \\ c &=& 1\end{array}\right.$$

Consequently,

$$\begin{aligned}
\int \frac{x^2+x+2}{(1+x^2)}dx &= \int 2-\frac{(x-1)}{1+x^2}dx \\
&= \int 2dx - \frac{1}{2}\int \frac{2x}{1+x^2}dx + \int \frac{1}{1+x^2}dx \\
&= 2x - \frac{1}{2}\ln(1+x^2) + \arctan(x) + c, \qquad \text{with } c \in \mathbb{R}.
\end{aligned}$$

4. Note that using the properties of the logarithm function, the integral can be rewritten as follows:

$$\int \ln\left(\frac{1}{\sqrt{1+x^2}}\right)dx = \frac{-1}{2}\int \ln\left(1+x^2\right)dx.$$

To calculate the latter, we use integration by parts. Let's take the following:

$$\left\{\begin{array}{lll} u' &=& 1 \\ v &=& \ln\left(1+x^2\right) \end{array}\right. \implies \left\{\begin{array}{lll} u &=& x \\ v' &=& \frac{2x}{1+x^2} \end{array}\right.$$

Thus,

$$\begin{aligned}
\int \ln\left(\frac{1}{\sqrt{1+x^2}}\right)dx &= \frac{-1}{2}\int \ln\left(1+x^2\right)dx = \frac{-1}{2}x\ln\left(1+x^2\right) + \int \frac{x^2}{1+x^2}dx \\
&= \frac{-1}{2}x\ln\left(1+x^2\right) + \int \left(1-\frac{1}{1+x^2}\right)dx \quad \text{(using Euclidean division)} \\
&= \frac{-1}{2}x\ln\left(1+x^2\right) + x - \arctan(x) + c, \qquad \text{with } c \in \mathbb{R}.
\end{aligned}$$

The Euclidean division of $\frac{x^2}{1+x^2}$

$$\begin{array}{rr|l}
 & x^2 & x^2+1 \\ \cline{3-3}
- & & 1 \\
 & x^2+1 & \\ \cline{1-2}
 & -1 &
\end{array}$$