1st order diff equ.

Exo 1:

- $y' - 2xy = 2x \Rightarrow y' = 2x + 2xy$

$\Rightarrow y' = 2x(1+y)$.

$\Rightarrow \frac{y'}{1+y} = 2x$ (separated variables)

$\Rightarrow \frac{dy}{1+y} = 2x\,dx$.

$\Rightarrow \ln(|1+y|) = x^2 + C \quad \Rightarrow |1+y| = Ke^{x^2}$.

$\Rightarrow \boxed{y = \pm K e^{x^2} - 1}$

- $y' = xy^2 - x - y^2 + 1 \Rightarrow y' = y^2(x-1) - (x-1)$.

$\Rightarrow y' = (y^2-1)(x-1)$.

$\Rightarrow \frac{y'}{y^2-1} = x-1 \Rightarrow \frac{dy}{y^2-1} = (x-1)dx$.

$\Rightarrow \int \frac{1}{y^2-1}\,dy = \int x-1\,dx$.

$\Rightarrow \int \frac{a}{y-1} + \frac{b}{y+1} = \frac{1}{2}x^2 - x + C \quad / \begin{matrix} a = \\ b = \end{matrix}$

$\Rightarrow \frac{1}{2}\int \frac{1}{y-1} + \frac{1}{y+1}\,dy = \frac{1}{2}x^2 - x + C$

$\Rightarrow \ln(|y^2-1|) = x^2 - 2x + C$

$\Rightarrow |y^2-1| = e^{x^2-2x+C}$.

$y^2 - 1 = \begin{cases} e^{x^2-2x+C} & \text{if } y \notin [-1,1] \\ -e^{x^2-2x+C} & \text{if } y \in [-1,1] \end{cases}$

$\Rightarrow y^2 = \begin{cases} e^{P_2} + 1 \\ e^{P_2} + 1 \end{cases}$

1

- $y' = \frac{y^2+1}{xy+y} \Rightarrow y' = \frac{y^2+1}{y} \times \frac{1}{x+1} \Rightarrow \frac{y}{y^2+1} y' = \frac{1}{x+1}$

$$\Rightarrow \frac{1}{2}\int \frac{2y}{y^2+1}\, dy = \int \frac{1}{x+1}\, dx.$$

$$\Rightarrow \frac{1}{2} \ln(y^2+1) = \ln(|x+1|) + C.$$

$$\Rightarrow \ln(y^2+1) = 2\ln(|x+1|) + C.$$

$$\Rightarrow y^2 + 1 = K(x+1)^2.$$

$$\Rightarrow y^2 = K(x+1)^2 - 1 \Rightarrow \boxed{y = \pm\sqrt{|K(x+1)^2 - 1|}}$$

- $(2xy^3 + 4x)y' = xy^2 + y^2 \Rightarrow 2x(y^3+2)y' = y^2(x+1)$.

$$\Rightarrow \frac{y^3+2}{y^2} y' = \frac{x+1}{2x} \Rightarrow \left(y + \frac{2}{y^2}\right) y' = \frac{1}{2} + \frac{1}{2x}.$$

$$\Rightarrow \int y + \frac{2}{y^2}\, dy = \int \frac{1}{2} + \frac{1}{2x}\, dx.$$

$$\Rightarrow \boxed{\frac{1}{2}y^2 - \frac{2}{y} = \frac{1}{2}x + \ln(\sqrt{|x|}) + C.}$$

- $y' = \frac{e^{x-y}}{1+e^x} \Rightarrow y' = \frac{e^x}{1+e^x} e^{-y} \Rightarrow e^y y' = \frac{e^x}{1+e^x} dx$

$$\Rightarrow \int e^y dy = \int \frac{e^x}{1+e^x} dx \Rightarrow e^y = \int \frac{1}{1+t} dt \,/\, t = e^x.$$

$$\Rightarrow e^y = \ln(1+t) + C. \Rightarrow y = \ln(|\ln(1+t)| + C)$$

we have $y(1) = 0$, $x = 1 \Rightarrow t = e$

$$0 = \ln(|\ln(1+e)| + C) \Rightarrow 1 = \ln(1+e) + C$$

$$\Rightarrow \boxed{C = 1 - \ln(1+e)}$$

$$y = \ln(|\ln(1+t) + 1 - \ln(1+e)|).$$

Q

- $y' = \dfrac{x^2y - y}{1+y} \Rightarrow y' = \dfrac{y}{1+y} * (x^2-1)$

$\Rightarrow \left(\dfrac{1+y}{y}\right) y' = x^2 - 1 \Rightarrow \int \dfrac{1}{y} + 1 \, dy = \int x^2 - 1 \, dx.$

$\Rightarrow \ln(|y|) + y = \frac{1}{3}x^3 - x + C.$

we have $y(3) = 1$ so $\underbrace{\ln(1)}_{=0} + 1 = 9 - 3 + C.$

$\Rightarrow \cancel{C = \ln(1)} \Rightarrow \boxed{C = -5}$

EXO2:

- $xy' - 2y = x \Rightarrow \begin{cases} xy_p' - 2y_p = 0 \\ y_{\cancel{G}} \end{cases}$

$xy' - 2y = 0 \Rightarrow xy' = 2y \Rightarrow \dfrac{y'}{y} = 2/x.$

$\Rightarrow \ln(|y|) = \ln(x^2) + C.$

$\Rightarrow \boxed{y = kx^2}$

Let put $K \equiv K(x)$ so $y = K(x)x^2$ and $y' = K'(x)x^2 + 2xK(x).$

thus, $x\left(K'x^2 + 2xK(x)\right) - 2Kx^2 = x.$

$K'(x)x^3 + \cancel{2x^2K} - \cancel{2Kx^2} = x \Rightarrow K'(x) = \dfrac{1}{x^2}$

$\Rightarrow \boxed{K(x) = -\dfrac{1}{x} + C.}$ consequently

$y_G = \left(-\dfrac{1}{x} + C\right) * x^2$ ie $\boxed{y_G = -x + Cx^2}$

- $xy' + 2y = \dfrac{\cos(x)}{x}.$

$xy' + 2y = 0 \Rightarrow xy' = -2y$

3

$$\Rightarrow \frac{y'}{y} = -\frac{2}{x} \Rightarrow \ln(y) = \ln\left(\frac{1}{x^2}\right) + c.$$

$$\Rightarrow \boxed{y = \frac{k}{x^2}} \quad \text{with } k = e^c \; / c \in \mathbb{R}.$$

Let $y = \frac{k(x)}{x^2} \Rightarrow y' = \frac{k'(x)\,x^2 - 2x\,k(x)}{x^{4}}.$

$$\Rightarrow \cancel{x}\left(\frac{x^2 k'(x) - 2x\,k(x)}{x^4}\right) + 2\frac{k}{x^2} = \frac{\cos(x)}{x}.$$

$$\frac{k'(x)}{\cancel{x}} - \cancel{\frac{2k(x)}{x^2}} + \cancel{\frac{2k(x)}{x^2}} = \frac{\cos(x)}{\cancel{x}}$$

$$\Rightarrow k'(x) = \cos(x). \Rightarrow \boxed{k(x) = \sin(x) + c}$$

so $\boxed{y = \frac{\sin(x) + c}{x^2}} \; / c \in \mathbb{R}.$

- $y' + \frac{2y}{x} = \frac{4}{x}. \Rightarrow y' + \frac{2y}{x} = 0$

$$\Rightarrow y' = \frac{-2y}{x} \Rightarrow \frac{y'}{y} = \frac{-2}{x}.$$

$$\Rightarrow \ln(y) = \frac{1}{x^2} + c \Rightarrow \boxed{y = \frac{k}{x^2}} \quad k \in \mathbb{R}_+ \; / k = e^c / c \in \mathbb{R}$$

$$y = \frac{k(x)}{x^2} \Rightarrow y' = \frac{x^2 k'(x) - 2x\,k(x)}{x^4}$$

$$\frac{x^2 k'(x) - 2x\,\cancel{k(x)}}{x^4} + \frac{2\,\cancel{k(x)}}{\cancel{x^3}} = \frac{4}{x}.$$

$$k'(x) = 4x \Rightarrow \boxed{k(x) = 2x^2 + c.}$$

$$\Rightarrow \boxed{y = \frac{2x^2 + c}{x^2}.}$$ we have $y(1) = 6$ so

$$6 = 2 + c \Rightarrow \boxed{c = 4}$$

$$\boxed{y = \frac{2x^2 + 4}{x^2}}$$

4

- $xy' - 2y = x^3 e^x$.

$xy' - 2y = 0 \Rightarrow xy' = 2y$.

$\Rightarrow \frac{y'}{y} = \frac{2}{x} \Rightarrow \ln(y) = \ln(x^2) + C$.

$\Rightarrow \boxed{y = Kx^2}$

we put $K \equiv K(x) \Rightarrow y = K(x)x^2$ and $y' = K'(x)x^2 + 2xK(x)$.

So,

$x(K'(x)x^2 + \cancel{2xK(x)}) - \cancel{2K(x)x^2} = x^3 e^x$.

$x^3 K'(x) = x^3 e^x \Rightarrow K'(x) = e^x \Rightarrow \boxed{K(x) = e^x + C}$ thus.

$\boxed{y = x^2(e^x + C)}$.

we have $y(1) = 0$ so $e + C = 0 \Rightarrow \boxed{C = -e}$

$\Rightarrow \boxed{y = x^2(e^x + e)}$

Ex03:

we say that a diff eq $y' = f(x,y)$ is homogeneous iff $f(\lambda x, \lambda y) = f(x,y)$ $\forall \lambda \in \mathbb{R}^*$.

- $y' = \frac{x^2 e^{y/x} + y^2}{xy}$, $f(x,y) = \frac{x^2 e^{y/x} + y^2}{xy}$.

$f(\lambda x, \lambda y) = \frac{\lambda^2 x^2 e^{\lambda y/\lambda x} + \lambda^2 y^2}{\lambda^2 xy} = \frac{x^2 e^{y/x} + y^2}{xy} = f(x,y) \Rightarrow$ homogen.

So to solve the equation we put $z = y/x \Rightarrow y = zx$.

$\Rightarrow y' = z'x + z$ after substitution we obtain:

$z'x + z = \frac{e^z + z^2}{z} \Rightarrow xz'z + z^2 = e^z + \cancel{z^2}$

5

$$\Rightarrow z e^{-z} z' = \frac{1}{x} \Rightarrow \underbrace{\int z e^{-z} dz}_{\text{by parts}} = \ln(x) + c.$$

$$\int z e^{-z} dz = -z e^{-z} + \int e^{-z} dz = -(z+1) e^{-z} + c.$$

$$u = z, \quad v' = e^{-z} \Rightarrow \begin{cases} u' = 1 \\ v = -e^{-z}. \end{cases}$$

so: $-(z+1) e^{-z} = \ln(x) + c.$

$$\Rightarrow \boxed{-\left(\frac{y}{x} + 1\right) e^{-y/x} = \ln(x) + c.}$$

$$y' = \frac{\sqrt{x^2 - y^2} + y}{x} \Rightarrow f(x,y) = \frac{\sqrt{x^2 - y^2} + y}{x}.$$

$$f(\lambda x, \lambda y) = \frac{\sqrt{\lambda^2 (x^2 - y^2)} + \lambda y}{\lambda x} = f(x,y) \checkmark$$

we put $\begin{cases} z = y/x \Rightarrow y' = x z' + z. \\ y = zx. \end{cases}$

$$x z' + z = \frac{\sqrt{x^2 - z^2 x^2} + z x}{x}$$

$$\Rightarrow x z' + \cancel{z} = \sqrt{1 - z^2} + \cancel{z}$$

$$\frac{z'}{\sqrt{1 - z^2}} = \frac{1}{x} \Rightarrow \arcsin(z) = \ln(|x|) + c$$

$$\rightarrow \boxed{z = \sin(\ln(|x|) + c)}$$

$$\Rightarrow \boxed{y = x \sin(\ln(|x|) + c)} \checkmark$$

8

- $y' = \dfrac{x^4+2y^4}{xy^3} \Rightarrow f(x,y) = \dfrac{x^4+2y^4}{xy^3}$.

$$f(\lambda x, \lambda y) = \frac{\lambda^4 x^4 + 2\lambda^4 y^4}{\lambda^4 x y^3} = f(x,y)$$

$$z = y/x \Rightarrow y = xz \Rightarrow y' = xz' + z$$

$$\cancel{x} z' + z = \frac{\cancel{x^4} + 2\cancel{x^4} z^4}{\cancel{x^4} z^3} = \frac{1+2z^4}{z^3}$$

$$\Rightarrow x z' + \cancel{z} = \frac{1}{z^3} + 2\cancel{z} \Rightarrow x z' = \frac{1+z^4}{z^3}$$

$$\Rightarrow z' \cdot \frac{z^3}{1+z^4} = \frac{1}{x} \Rightarrow \frac{1}{4} \frac{4z^3}{1+z^4} \cancel{z} z' = \frac{1}{x}$$

$$\Rightarrow \ln(1+z^4) = \ln(x^4) + C$$

$$\Rightarrow 1+z^4 = kx^4 \Rightarrow z^4 = kx^4 - 1$$

$$\Rightarrow z = \sqrt[4]{|kx^4-1|}$$

$$\Rightarrow \boxed{y = x\left(|kx^4-1|\right)^{\frac{1}{4}}}$$

9

Exo4:

• $y' + \frac{y}{x} = 3x^2y^2$

$\Rightarrow y^{-2}y' + \frac{y^{-1}}{x} = 3x^2$ — (*)

we put $z = y^{-1} \Rightarrow z' = -y^{-2}y'$

so (*) $\Rightarrow -z' + \frac{z}{x} = 3x^2$ — (**)

$-z' + \frac{z}{x} = 0 \Rightarrow -z' = -\frac{z}{x}$

$\Rightarrow \frac{z'}{z} = \frac{1}{x}$

$\Rightarrow \ln(z) = \ln(x) + c$

$\Rightarrow \boxed{z = Kx}$

let's assume that $K = K(x)$

then $z = K(x)x$ and $z' = K'(x)x + K(x)$

$\Rightarrow -(K'(x)x + K(x)) + \frac{K(x)x}{x} = 3x^2$

$\Rightarrow -xK'(x) \cancel{-K(x)} \cancel{+K(x)} = 3x^2$

$\Rightarrow K'(x) = 3x \Rightarrow \boxed{K(x) = \frac{3}{2}x^2 + C}$

$\Rightarrow z_g = \left(\frac{3}{2}x^2 + c\right)x$

we have $z = \frac{1}{y} \Rightarrow \boxed{y = \frac{1}{z_g} = \frac{1}{x\left(\frac{3}{2}x^2 + c\right)}}$

• $y' - 4y = 2e^{x}\sqrt{y} \Rightarrow y^{-1/2}y' - 4y^{1/2} = 2e^{x}$

we put $z = y^{1/2} \Rightarrow z' = \frac{1}{2} y^{-1/2} y'$

$\Rightarrow 2z' - 4z = 2e^{x}$

$\Rightarrow \boxed{z' - 2z = e^{x}}$ —— (*)

$z' - 2z = 0 \Rightarrow z' = +2z \Rightarrow z'/z = +2$

$\Rightarrow \ln(z) = +2x + c$

$\Rightarrow \boxed{z_H = K e^{+2x}}$

we assume that $K = k(x)$ then $z_G = k(x) e^{+2x}$

and $z'_G = k'(x)e^{2x} + 2k(x)e^{2x}$

(*) $\Rightarrow k'(x)e^{+2x} + \cancel{2k(x)e^{+2x}} - \cancel{2k(x)e^{+2x}} = 2e^{x}$

$\Rightarrow k'(x) = 2e^{-x} \Rightarrow \boxed{k(x) = -2e^{-x} + c}$

$\Rightarrow z_G = (c - 2e^{-x})e^{2x}$

$\boxed{z_G = (ce^{2x} - 2e^{x})}$

we have $z = \sqrt{y} \Rightarrow y = z^2 \Rightarrow y_G = (ce^{2x} - 2e^{x})^2$.

• $y' - \frac{3}{4}y = (9x-3)y^5 \Rightarrow y^{-5}y' - \frac{3}{4}y^{-4} = 9x-3$ —(*)

we put $z = y^{-4} \Rightarrow z' = -4y^{-5}y'$

$(*) \Rightarrow -\frac{1}{4}z' - \frac{3}{4}z = 9x-3$ —(**)

$-\frac{1}{4}z' - \frac{3}{4}z = 0 \Rightarrow z' + 3z = 0 \Rightarrow z'/z = -3$

$\Rightarrow \ln(z) = -3x + c$

$\Rightarrow \boxed{z = Ke^{-3x}}$

$z = K(x) \Rightarrow z' = K'(x)e^{-3x} - 3K(x)e^{-3x}$

$(**) \Rightarrow -\frac{1}{4}K'(x)e^{-3x} + \cancel{\frac{3}{4}K(x)e^{-3x}} - \cancel{\frac{3}{4}K(x)e^{-3x}} = 9x-3$

$\Rightarrow K'(x) = (-36x+12)e^{3x}$

$\Rightarrow K(x) = (-12x+4)e^{3x} + \int 12e^{3x}\,dx$

$\begin{cases} u = -36x+12 \\ v' = e^{3x} \end{cases} \Rightarrow \begin{cases} u' = -36 \\ v = \frac{1}{3}e^{3x} \end{cases}$

$\Rightarrow \boxed{K(x) = (-12x+4)e^{3x} + 4e^{3x} + C}$

$\Rightarrow z_g = \left[(-12x+8)e^{3x} + C\right] \times e^{-3x}$

$\boxed{z_g = (-12x+4) + Ce^{-3x}}$

we have $z = \frac{1}{y^4} \Rightarrow y^4 = \frac{1}{z} \Rightarrow y_g = \left(\frac{1}{z}\right)^{\frac{1}{4}}$.

• $3y' + \frac{3}{x}y = 2x^2y^4$.

$$3y^{-4}y' + \frac{3y^{-3}}{x} = 2x^2 \Rightarrow z = y^{-3} \Rightarrow z' = -3y^{-4}y'$$

$$\Rightarrow \boxed{-z' + \frac{3z}{x} = 2x^2}$$

$$-z' + \frac{3z}{x} = 0 \Rightarrow z' = \frac{3z}{x} \Rightarrow \frac{z'}{z} = \frac{3}{x}$$

$$\ln(z) = 3\ln(x) + C \Rightarrow \boxed{z_H = Kx^3}$$

$$K = K(x) \Rightarrow z_G = K'(x)x^3 + 3K(x)x^2$$

$$\Rightarrow -K'(x)x^3 \cancel{- 3K(x)x^2} + \cancel{3\frac{K(x)x^3}{x}} = 2x^2$$

$$-\cancel{x^3}K'(x) = 2\cancel{x^2} \Rightarrow K'(x) = \frac{-2}{x}$$

$$\Rightarrow K(x) = -2\int \frac{1}{x}\,dx = \boxed{-2\ln(|x|) + C = K(x)}$$

$$\Rightarrow \boxed{z_G = -2x^3\ln(|x|) + Cx^3}$$

$$z = \frac{1}{y^3} \Rightarrow \frac{1}{z} = y^3 \Rightarrow y = \frac{1}{\sqrt[3]{z}}$$

$$\Rightarrow \boxed{y_G = \frac{1}{\sqrt[3]{(-2\ln(|x|)+C)x^3}} = \frac{1}{x\sqrt{-2\ln(|x|)+C}}}$$