University of Biskra
Mathematics Department
Module: Analysis 2

First year license
2023/2024

Worksheet N°1

Exercise 1 Calculate the derivatives of the following univariate functions:

1. $f(x,y) = e^{x^2+y}$, $\quad g(x,y) = x^2 + y^2$, $\quad h(x,y) = x^y$.
2. $\arcsin\left(\frac{x}{\sqrt{a}}\right)$, $\quad \arccos\left(\frac{x}{\sqrt{a}}\right)$, $\quad \arctan\left(\frac{x}{a}\right)$, with $a > 0$.

Exercise 2 Let's consider the following functions:

$$\begin{aligned}
f_1(x) &= ln\left(\frac{\alpha x + \beta}{ax + b}\right). \\
f_2(x) &= \frac{\alpha}{2a}\ \ln(ax^2 + b) + \frac{\beta}{\sqrt{ab}}\ \arctan\left(\sqrt{\frac{a}{b}}\ x\right),\ \text{with } a, b > 0. \\
f_3(x) &= \frac{\alpha}{\sqrt{a}}\ \arcsin\left(\sqrt{\frac{a}{b}}\ x\right) - \frac{\beta}{a}\ \sqrt{b - ax^2},\ \text{with } a, b > 0. \\
f_4(x) &= \sum_{k=0}^{q} \frac{(-1)^k\ C_q^k}{(n+2k+1)} \cos(x)^{n+2k+1}, \qquad \text{with } n, k, q \in \mathbb{N} \text{ and } C_q^k = \frac{q!}{(q-k)!\ k!}.
\end{aligned}$$

1. Give the derivatives of the functions f_1, f_2 and f_3 in their simplified form.
2. Check that f_4' can be expressed as a product of $\cos^n$ and $\sin^m$ (n and m are two natural numbers).
3. Check that the proposition of the second question remains true when we replace *cos* with *sin* in the expression of f_4.

Exercise 3 Considering the polynomial $P_2(x) = x^2 + ax + b$, with $a, b \in \mathbb{R}$. Show that:

1. If $a^2 - 4b \geq 0$ then $P_2(x)$ can be rewritten $P_2(x) = (x + A)^2 - B^2$.
2. If $a^2 - 4b < 0$ then $P_2(x)$ can be rewritten $P_2(x) = (x + A)^2 + B^2$.

Exercise 4 Determine the expression of the function f that must be differentiated to obtain the following:

$$\begin{aligned}
f'(x) &= \cos(x), & f'(x) &= x + 2x^{\sqrt{2}} + 1, & f'(x) &= \tfrac{x+1}{1+x^2}, \\
f'(x) &= \tfrac{x+1}{\sqrt{1-x^2}}, & f'(x) &= e^{\frac{x}{2}} + 1
\end{aligned}$$

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Correction

Solution of the Exercise 1

1. The objective of the first part of this exercise is to remind the student that in the context of differentiation (and integration) before any action, it is first necessary to identify the variables and constants involved in the expression of f. In addition, through these 03 examples, we show them that the differentiation according to x and the differentiation according to y will not provide us with the same result.

   From the statement of the exercise we know that the functions in question are univariate but what is not clear is that the variable. Indeed, in the exercise, it was not specified whether the variable is x or if it is y.

   First case If we assume that "x is the variable and y is a constant", then we will have the following:

$$f'(x,y) = \left(e^{x^2+y}\right)' = 2xe^{x^2+y}, \quad g'(x,y) = \left(x^2+y^2\right)' = 2x, \quad h'(x,y) = (x^y)' = (y-1)x^{y-1}.$$

   Second case If we assume that "y is the variable and x is a constant", then we will have the following:

$$f'(x,y) = \left(e^{x^2+y}\right)' = e^{x^2+y}, \quad g'(x,y) = \left(x^2+y^2\right)' = 2y, \quad h'(x,y) = (x^y)' = \ln(x)x^y.$$

2.

$$\begin{aligned}
(\arcsin(f(x)))' &= \frac{f'(x)}{\sqrt{1-f(x)^2}} = \frac{\frac{1}{\sqrt{a}}}{\sqrt{1-x^2/a}} = \frac{1}{\sqrt{a}\sqrt{1-x^2/a}} = \frac{1}{\sqrt{a-x^2}}.\\
(\arccos(f(x)))' &= \frac{-f'(x)}{\sqrt{1-f(x)^2}} = \frac{-1}{\sqrt{a-x^2}}.\\
(\arctan(f(x)))' &= \frac{f'(x)}{1+f(x)^2} = \frac{\frac{1}{a}}{1+x^2/a^2} = \frac{a}{a^2(1+x^2/a^2)} = \frac{a}{a^2+x^2)}.
\end{aligned}$$

Solution of the Exercise 2

$$\begin{aligned}
f_1'(x) &= ln\left(\frac{\alpha x+\beta}{ax+b}\right)' = \frac{\left(\frac{\alpha x+\beta}{ax+b}\right)'}{\left(\frac{\alpha x+\beta}{ax+b}\right)} = \frac{\frac{\alpha ax+\alpha b-\alpha ax-\beta b}{(ax+b)^2}}{\left(\frac{\alpha x+\beta}{ax+b}\right)} = \frac{\alpha b-\beta b}{(ax+b)^2}\ \frac{ax+b}{\alpha x+\beta} = \frac{\alpha b-\beta b}{(\alpha x+\beta)\,(ax+b)}.\\
f_2'(x) &= \left(\frac{\alpha}{2a}\ \ln(ax^2+b)\right)' + \left(\frac{\beta}{\sqrt{ab}}\ \arctan\left(\sqrt{\frac{a}{b}}\ x\right)\right)' = \frac{\alpha}{2a}\left(\frac{2ax}{ax^2+b}\right) + \frac{\beta}{\sqrt{ab}}\left(\frac{\sqrt{a/b}}{1+\frac{a}{b}x^2}\right),\\
&= \frac{\alpha x+\beta}{ax^2+b}\\
f_3'(x) &= \left(\frac{\alpha}{\sqrt{a}}\ \arcsin\left(\sqrt{\frac{a}{b}}\ x\right)\right)' - \left(\frac{\beta}{a}\ \sqrt{b-ax^2}\right)' = \frac{\alpha}{\sqrt{a}}\left(\frac{\sqrt{a/b}}{\sqrt{1-\frac{a}{b}x^2}}\right) - \frac{\beta}{a}\left(\frac{-2ax}{2\sqrt{b-ax^2}}\right)\\
&= \frac{\alpha+\beta x}{\sqrt{b-ax^2}}.
\end{aligned}$$

$$
\begin{aligned}
f_4'(x) &= \left(\sum_{k=0}^{q} \frac{(-1)^k\ C_q^k}{(n+2k+1)} \cos(x)^{n+2k+1}\right)' = \sum_{k=0}^{q}\left(\frac{(-1)^k\ C_q^k}{(n+2k+1)} \cos(x)^{n+2k+1}\right)' \\
&= \sum_{k=0}^{q}(-1)^k\ C_q^k \sin(x)\cos(x)^{n+2k} = \sin(x)\cos(x)^n \left(\sum_{k=0}^{q} C_q^k \left(-\cos^2(x)\right)^k\right) \\
&= \sin(x)\cos(x)^n(1-\cos^2(x))^q, \quad \textit{as} \sum_{k=0}^{q} C_q^k \left(-\cos^2(x)\right)^k \ \textit{ is the binomial expansion} \\
&= \sin(x)\cos(x)^n \sin^{2q}(x) = \cos(x)\sin^m(x), \quad \textit{with } m = 2q+1.
\end{aligned}
$$

Solution of the Exercise 3

$$
\begin{aligned}
P_2(x) &= x^2 + ax + b \\
&= x^2 + 2\left(\frac{a}{2}\right)x + \left(\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 + b \\
&= \left(x + \left(\frac{a}{2}\right)\right)^2 - \left(\frac{a}{2}\right)^2 + b
\end{aligned}
$$

From this latest formula it claire that $A = \frac{a}{2}$ and $B = \begin{cases} \sqrt{b - \left(\frac{a}{2}\right)^2}, & \text{if } a^2 - 4b < 0; \\ \sqrt{\left(\frac{a}{2}\right)^2 - b}, & \text{if } a^2 - 4b \geq 0. \end{cases}$ hence $P_2(x) = \begin{cases} (x+A)^2 + B^2 & \text{with } A = \frac{a}{2} \text{ and } B = \sqrt{b - \left(\frac{a}{2}\right)^2}, \quad \text{if } a^2 - 4b < 0; \\ (x+A)^2 - B^2 & \text{with } A = \frac{a}{2} \text{ and } B = \sqrt{\left(\frac{a}{2}\right)^2 - b}, \quad \text{if } a^2 - 4b \geq 0; \end{cases}$

Solution of the Exercise 4 Let's consider c a real constant in all of the present exercise.

1. we have $f'(x) = \cos(x)$ and as $(\sin(x) + c)' = \cos(x)$ then $f(x) = \sin(x) + c$.
2. we have $f'(x) = x + 2x^{\sqrt{2}} + 1$ and as $\left(\frac{1}{2}x^2\right)' = x$, $\left(\frac{2}{\sqrt{2}+1}x^{\sqrt{2}+1}\right)' = 2x^{\sqrt{2}}$, $(x)' = 1$ and $c' = 0$ then $f(x) = \frac{1}{2}x^2 + \frac{2}{\sqrt{2}+1}x^{\sqrt{2}+1} + x + c$.
3. we have $f'(x) = \frac{x+1}{1+x^2} = \frac{x}{1+x^2} + \frac{1}{1+x^2}$ and as $\left(\frac{1}{2}\ln(1+x^2)\right)' = \frac{x}{1+x^2}$ and $(\arctan(x)) = \frac{1}{1+x^2}$ then $f(x) = \frac{1}{2}\ln(1+x^2) + \arctan(x) + c$. (To solve this exercise we can also use the result of the second exercise when we put $a = b = \alpha = \beta = 1$).
4. we have $f'(x) = \frac{x+1}{\sqrt{1-x^2}} = \frac{x}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}$ ans as $\left(-\sqrt{1-x^2}\right)' = \frac{x}{\sqrt{1-x^2}}$ and $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ then $f(x) = \arcsin(x) - \sqrt{1-x^2} + c$.
5. we have $f'(x) = e^{\frac{x}{2}} + 1$ and as $\left(2e^{\frac{x}{2}}\right)' = e^{\frac{x}{2}}$ and $(x)' = 1$ then $f(x) = 2e^{\frac{x}{2}} + x + c$.