Nonlinear Models

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1. Generalities about Nonlinear Models

1) Definition of Nonlinear Models:

- A nonlinear model is one where the relationship between the dependent variable (Y) and one or more independent variables (X) is not a straight line (not linear).
- It can take various forms: quadratic, exponential, logarithmic, etc

1. Generalities about Nonlinear Models

2) Mathematical Form:

- **Examples:**

✓Exponential Growth model: $Y = \beta_0 \cdot e^{\beta_1 X}$

✓Logarithmic model: Y= $\beta_0 + \beta_1 \ln(X)$

✓Power Law model: $Y = \beta_0 X^{\beta_1}$

✓Logit model : $\boldsymbol{P(Y = 1/X) = \frac{1}{1+e^{-(\beta_0+\beta_1 X_1+\beta_2 X_2 \cdots\cdots+\beta_k X_k)}}}$

✓Probit model

3. Estimation method

1) **Linearizable Models:**

- Transform into linear form, then use ordinary least squares (OLS) to estimate parameters.
- **Nonlinearizable Models:** Cannot be transformed into a linear form.
- Must be estimated using specialized nonlinear regression techniques. such as:
- **(b) Maximum Likelihood Estimation (MLE):** Used when the model is probabilistic.
- **(c) Generalized Method of Moments (GMM):** Useful when specific assumptions about distributions cannot be made.

4. Examples of nonlinear models

1) Implement a Nonlinear Model Using Eviews:

1. Import data into EViews.
2. Specify the Nonlinear Equation:
   - Go to **Quick** → **Estimate Equation**.
   - Enter the equation:
3. Select the **Nonlinear Least Squares (NLS)** estimation method.
4. Run the Model: EViews will estimate the parameters b0 and b1.
5. Analyze Results: Look at the estimated coefficients and goodness of fit statistics (R-squared, residual plots).

4. Examples of nonlinear models

- **Example 1:** A company observes that sales increase over time at a increasing rate.
- The relationship can be modeled as:
- $sales = \beta_0 . e^{\beta_1 . time} + \varepsilon$, where:
- β_0: Initial sales.
- β_1: Growth rate. And *time*: Time in months.
- The following table gives the simulated Data:

4. Examples of nonlinear models

Time (Months)	Sales (Units)
1	50
2	75
3	130
4	200
5	350

- Enter the equation:

sales = c(1) * exp(c(2)*time)

Eviews result

Dependent Variable: SALES
Method: Least Squares (Gauss-Newton / Marquardt steps)
Date: 01/23/25 Time: 16:54
Sample: 1 5
Included observations: 5
Convergence achieved after 22 iterations
Coefficient covariance computed using outer product of gradients
SALES = C(1) * EXP(C(2)*TIME)

	Coefficient	Std. Error	t-Statistic	Prob.
C(1)	27.35122	2.175883	12.57017	0.0011
C(2)	0.508026	0.017503	29.02484	0.0001

R-squared	0.997823	Mean dependent var	161.0000
Adjusted R-squared	0.997098	S.D. dependent var	120.3329
S.E. of regression	6.482876	Akaike info criterion	6.865380
Sum squared resid	126.0830	Schwarz criterion	6.709155
Log likelihood	-15.16345	Hannan-Quinn criter.	6.446088
Durbin-Watson stat	2.881891		

Interpretation

- The coefficient c(2) or (β_1) represents the **growth rate** of sales.
- Specifically:If c(2)>0, sales grow exponentially over time.
- If c(2)<0, sales decline exponentially over time.
- Since c(2)=0.508, this indicates a **positive exponential growth rate**. Sales are increasing
- A p-value of 0.0001 (very small) means that there is **strong evidence** against the null hypothesis (H0: β_1=0).
- This suggests that β_1 is significantly different from zero, and the exponential growth is highly statistically significant.

Interpretation

- Since $\beta_1 = 0.508$, this indicates a **positive exponential growth rate**. Sales are increasing over time, and the magnitude of 0.508 shows the speed of growth.
- **Growth Interpretation**: The sales increase by approximately 50.8% per unit of time (assuming the time variable is measured in consistent units, e.g., days, months, years).

Interpretation

- β_0 : is the value of Y when X=0.
- Sign of β_1 :If β_1 > 0, Y grows exponentially as X increases.
- If β_1 <0, Y decreases exponentially as X increases.
- Positive β_1 : Y grows faster as X increases.
- Negative β_1 : Y diminishes faster as X increases.

4. Examples of nonlinear models

- EViews outputs:
- β_0=*27.35*, β_1=*0.508*
- **Prediction Formula:**
- $sales = 27.35 \cdot e^{0.508time}$
- For Time = 6 months:
- $sales = 27.35 \cdot e^{0.508 * 6}$ ≈ 576.16

4. Examples of nonlinear models

- **Example 2: Logarithmic Model, which assumes a diminishing return effect, where increases in the independent variable lead to progressively smaller increases in the dependent variable.** In the following table we have data about advertising expenditure (adv) and sales (sales) data, to study the diminishing effect of advertising expenditure on sales. $sales = \beta_0 + \beta_1 \ln(adv)$

Obsno	adv	sales
1	10	5.3
2	20	6.8
3	30	7.6
4	40	8.1
5	50	8.5

4. Examples of nonlinear models

- Import the data into EViews, naming columns X and Y.
- Go to **Quick → Estimate Equation**.
- Enter the model:

```
sales = c(1) + c(2) * log (adv)
```

  - c(1) represents β_0.
  - c(2) represents β_1 .

Run the Estimation:

- Click **OK** to estimate the parameters using Ordinary Least Squares (OLS).

Eviews Results

Dependent Variable: SALES
Method: Least Squares (Gauss-Newton / Marquardt steps)
Date: 01/10/25   Time: 22:47
Sample: 1 5
Included observations: 5
SALES = C(1) + C(2) * LOG (ADV)

	Coefficient	Std. Error	t-Statistic	Prob.
C(1)	0.772338	0.179835	4.294699	0.0232
C(2)	1.990030	0.054343	36.61986	0.0000

R-squared	0.997768	Mean dependent var	7.260000
Adjusted R-squared	0.997024	S.D. dependent var	1.266096
S.E. of regression	0.069071	Akaike info criterion	-2.218193
Sum squared resid	0.014312	Schwarz criterion	-2.374418
Log likelihood	7.545483	Hannan-Quinn criter.	-2.637485
F-statistic	1341.014	Durbin-Watson stat	1.522667
Prob(F-statistic)	0.000045		

interpretation

- The p-value of 0.0000 indicates that the coefficient c(2) is **statistically significant**.
- This provides very strong evidence against the null hypothesis (H0:β_1 =0), confirming that there is a strong relationship between advertising and sales.

interpretation

- The coefficient $\beta_1 = 1.99$ means that a 1% increase in advertising leads to a 1.99% increase in sales.
- However, this is a proportional (percentage) increase, not an absolute unit increase.
- The absolute gain gets smaller relative to the increase in advertising, because:
- $\frac{d(sales)}{d(adv)} = \frac{1.99}{adv}$ The marginal effect (extra sales per additional unit of adv) decreases as advertising increases.

interpretation

- The increase in sales per extra **one unit** of advertising is 1.99/adv, which decreases as adv increases.
- The marginal effect 1.99/adv explains diminishing returns, showing that the effect of one extra unit of advertising gets smaller as advertising increases.
- **The diminishing effect comes from taking ln(adv) in the model.**
- **Each additional unit of advertising has a smaller effect than the previous unit.**
- **A 1% increase in advertising has the same proportional effect (1.99% of current sales), but the absolute gain (**1.99/adv) **gets smaller as advertising increases.**

Interpretation

EViews outputs:

- β_0=*0.773*, β_1=*1.99*
- **Prediction Formula:**

$$sales = 0.773 + 1.99\ln(adv)$$

For Advertising expenditure of 75

Sales= 0.773 + 1.99 ln(75) = 9.364

For Advertising expenditure of 90

Sales= 0.773 + 1.99 ln(90) = 9.72.

For Advertising expenditure of 105

Sales= 0.773 + 1.99 ln(105) = 10.03

The increase in the advertising by 15 unit from (75 to 90) increase the sales by 0.36

And the increase of advertising expenditure by 15 unit from 90 to 105; the sale increased by 0.31

Interpretation

- β_0: is the value of Y (when ln(X)=0 or X=1).
- Sign of β_1 :Positive β_1 : Y increases as X increases but at a decreasing rate.
- Negative β_1 : Y decreases as X increases.

4. Example of nonlinear model

- **The Power Law model $Y = \beta_0 X^{\beta_1}$** is a type of mathematical relationship where a dependent variable Y is proportional to a power of the independent variable X. It's a nonlinear relationship commonly used in fields like physics, economics, biology, and social sciences to describe scaling behavior and proportional changes.
- We use this model in economy to describe scaling relationships between inputs and outputs.

4. Example of nonlinear model

- **Key Components of the Power Law Model**

1.Mathematical Form:

$Y = \beta_0 X^{\beta_1}$

Y: Dependent variable (output).

X: Independent variable (input).

β_0: Proportionality constant, scaling factor (the value of Y when X=1).

β_1: Exponent or scaling factor that determines how Y changes with X.

4. Example of nonlinear model

- **Log-Linear Transformation**:
- To linearize the equation for estimation, take the natural logarithm of both sides: $\ln(Y) = \ln(\beta_0)+\beta_1\ln(X)$.
- Let $Y'=\ln(Y)$, $X'=\ln(X)$, and $\ln(\beta_0)= \beta_0'$.
- The model becomes: $Y'=\beta_0'+\beta_1X'$. This is now a linear regression model.

4. Example of nonlinear model

- **Interpretation of Parameters**
- β_0:The **scaling factor**. Represents the base value of Y when X=1 (since $X^{\beta_1}=1$ when X=1).
- β_1:The **elasticity** or exponent. It describes how Y changes as X increases:
- If $\beta_1>1$, Y increases faster than X (increasing returns to scale).
- If $\beta_1=1$, Y changes proportionally with X (linear relationship).
- If $0<\beta_1<1$, Y increases at a diminishing rate (diminishing returns to scale).
- If $\beta_1<0$, Y decreases as X increases (inverse relationship).

4. Example of nonlinear model

- Numerical example: A researcher studies the relationship between the area of a city (X) in square kilometers and its population (Y) in thousands. The relationship between the area and population is believed to follow the Power Law Model:
- The researcher collects the following data:

City	Area Mk2	Population (thaousand)
A	1	1
B	5	10
C	10	23
D	20	54
E	50	164
F	100	376
G	200	865
H	500	2599
I	1000	5971
J	2000	13719

4. Example of nonlinear model

- **Tasks**

1. Transform the nonlinear model into a linearized form suitable for estimation.
2. Estimate the parameters β_0 and β_1 using ordinary least squares (OLS). Predict the population when the city area is X=90 sq. km.

- **Step 1: Enter Data in EViews**

1. Open **EViews** go file choose import data.

- **Step 2: Transform Data**

1. Generate new variables:
   - Log of population: series log_population=log(lpopulation)
   - Log of area: series log_area: log(area)

- **Step 3: Estimate the Log-Log Model**

1. Open the **Equation Estimation** window.
2. Enter the regression equation:
   - Log(population) c log(area)
3. Choose **Ordinary Least Squares (OLS)** and click **OK**.

- **Step 4: Interpret Results**
- After running the regression, EViews will output estimates for:
- $\ln Y = \ln \beta_0 + \beta_1 \ln X$
- **Intercept (β_0)**: Take **exponential (exp)** to get it's true interpretation.
- **Slope (β_1)**: This is the **elasticity** Y with respect to X (percentage change in population for a 1% change in area).
- **R^2**: Shows how well the model explains the variation in population.

E-views results

Dependent Variable: LOG(POPULATION)
Method: Least Squares
Date: 02/01/25 Time: 12:19
Sample: 1 10
Included observations: 10

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.215036	0.064840	3.316410	0.0106
LOG(AREA)	1.234192	0.013652	90.40087	0.0000

R-squared	0.999022	Mean dependent var	5.330334
Adjusted R-squared	0.998900	S.D. dependent var	3.018479
S.E. of regression	0.100121	Akaike info criterion	-1.588022
Sum squared resid	0.080193	Schwarz criterion	-1.527505
Log likelihood	9.940111	Hannan-Quinn criter.	-1.654409
F-statistic	8172.318	Durbin-Watson stat	1.298601
Prob(F-statistic)	0.000000		

Interpreting the Results

we get the following estimated equation:

- $\ln Y = 4.805 + 0.031 \ln X$

2. This means:

- $\beta_0 = e^{0.215} \approx 1.240$ (the base level of population when area = 1 km^2).
- $\beta_1 = 1.234$:
- Since the p-value related to β_1 is less than the significance level , so the area has a significance level on the population, and $\beta_1 > 1$, Y increases faster than X (increasing returns to scale).
- **1% increase** in area leads to a **1.234% increase** in population, following the elasticity property of log-log models.

Make prediction

- This means ou regression is $Y = 1.240X^{1.234}$
  - If is 90 then:
  - $Y = 1.240 * 90^{1.234} \approx 1.240 * 113.16 \approx 140.72$

Summary

a) In a **logarithmic model**, the dependent variable increases at a decreasing rate as the independent variable increases, and the model takes the form

$$Y = \beta_0 + \beta_1 \ln X.$$

a) In an **exponential growth model**, the dependent variable grows at a constant percentage rate relative to its current value, and the model takes the form $Y = \beta_1 e^X$.

b) In a **power law model**, the relationship follows a proportional scaling rule, meaning a percentage increase in X leads to a fixed percentage change in Y, and the model takes the form $Y = \beta_0 X^{\beta_1}$.

4. Examples of nonlinear models

- **Example 3:**

Time (X_t)	Population (Y_t)
1	100
2	200
3	320
4	520
5	780

Formulate a nonlinear model presenting this relationship?
Estimates the parameter, and interprete the result? Predict thee value of Y when X =12

Example 2

Month	Advertising Spend (Thousands)	Sales Revenue (Millions)
1	5	0.8
2	10	1.5
3	20	2.9
4	30	4.3
5	40	5.6
6	50	7.2
7	60	8.4
8	70	9.6
9	80	11.1
10	100	13.2

- **Example 2:** company is studying the relationship between **Advertising Spend** (in thousands of dollars) and **Sales Revenue** (in millions of dollars). The company wants to estimate how changes in advertising spend affect sales revenue, and they are considering different models to analyze this relationship.Given the data in the above table:
- Based on the data provided, which **model** do you think is more suitable for estimating the relationship between **Advertising Spend** and **Sales Revenue**? Justify your choice?
- Estimate the parameters of the model you selected intercept ($\beta 0$) and slope ($\beta 1$).
- Interpret the meaning of the **intercept** and **slope** in the context of the relationship between advertising spend and sales revenue.
- Using the model you estimated, predict the **Sales Revenue** for an advertising spend of **$75,000**.

Examples of nonlinear models

- **Example 3:** The following table shows the **years of experience of employees** and their **productivity score (out of 100)** in a company:
- Based on the data provided, which **model** do you think is more suitable for estimating the relationship between years of experience of employées and their productivity score?
- Estimate the parameters of the model you selected intercept (β0) and slope (β1).
- Interpret the meaning of the **intercept** and **slope** in the context of the relationship between years of experience of employees and their productivity.
- Suppose an employee has **8 years of experience**. Use your estimated model to predict their expected productivity score.