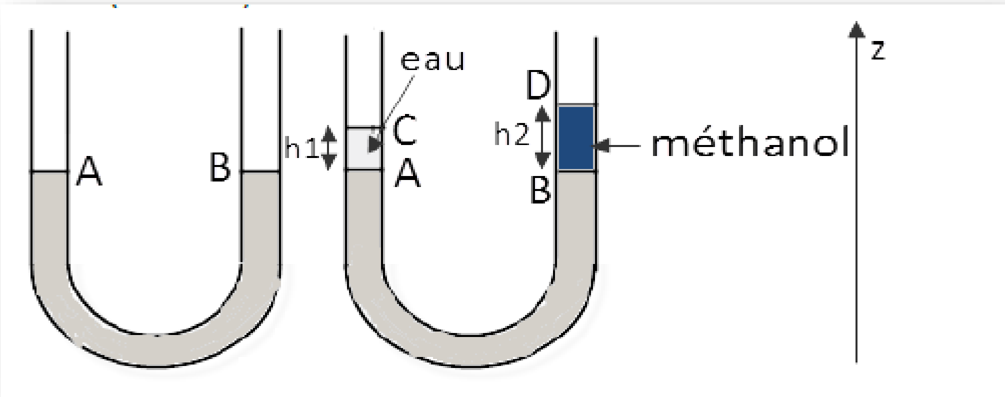
Tutorial N°5: Exercises on Pascal's law and Archimedes thrust. (Hydrostatic)

Exercise 5.1:

To determine the volume mass of ethanol $\rho_{\text{ ethanol}}$, glycerin is introduced into a U tube. In the left branch, water of density $\rho_{\text{ water}} = 1000\ kg.m^{-3}$ is poured over a height $h_1 = 10cm$, which causes a difference in level between the points A and B . To bring the points A and B back to the same height, methanol is poured over a height $h_2 = 12.5\ cm$ (diagram).



1. Write the fundamental hydrostatic relationship for the three fluids.

2. Deduce the volumic mass (density) of ethanol $\rho_{\text{ ethanol}}$

Solution

1- The fundamental hydrostatic relationship for the three fluids:

Glycerin: $P_A - P_B = 0$ (1)

Water: $P_A - P_C = \rho_{water}.h_1 . g$ (2)

Methanol $P_B - P_D = \rho_{\text{Methanol}}.h_2 . g$ (3)

2- So we have:

from (1) $P_A = P_B$

from (2) $P_A = P_C + \rho_{water}.h_1 . g$

from (3) $P_B = P_D + \rho_{\text{Methanol}}.h_2 . g$

We also have: $P_C = P_D = P_{atm}$

from where: $\rho_{\text{Methanol}}.h_2 .g = \rho_{water}.h_1 .g$

$$\rho_{\text{Methanol}} = \frac{\rho_{water}.h_1}{h_2} = \frac{1000 \times 10}{12.5} = 800Kg.m^{-3}$$

Exercise 5.2:

A hollow steel sphere of density $\rho_{steel} = 7600\ Kg.m^{-3}$ and radius $r = 20cm$ and thickness$e = 5mm$.

1- Determine the weight of this sphere.

2- Determine the Archimedes' thrust that would be exerted on this sphere if it were totally immersed in water.

3- Determine the force that Archimedes would exert on this ball if it were completely submerged in water.

4- Could this sphere float on the surface of water? If yes, then what is the fraction of its submerged volume?

Solution

1-The volume of the hollow sphere V_{HC} is

Volume of the hollow sphere V_{HC} = Volume of the sphereV_S − vacuum volume V_V

$$V_{HC} = \frac{4}{3}\pi r^3 - \frac{4}{3}\pi(r-e)^3 = \frac{4}{3}\pi[r^3 - (r-e)^3] = \frac{4}{3}\pi[(0.2)^3 - (0.2 - 0.008)^3]$$

$$V_{HC} = 3.86 \times 10^{-3} m^3$$

The weight P_{HC} of the hollow sphere is then:

$$P_{HC} = mg = \rho_{steel}.V_{HC}.g = 7600 \times 3.86 \times 10^{-3} \times 9.81 = 287.79N$$

2. The Archimedes thrust for the totally submerged ball is the weight of the displaced volume of the water therefore:

$$\boldsymbol{\pi = \rho_{water}.V_S.g = \rho_{water}.\frac{4}{3}\pi r^3.g = 1000.}\frac{4}{3}\pi.(0.2)^3.9.81 = 328.74N$$

The ball will float because the Archimedes π thrust is greater than its weightP_{HC}:

$$\pi > P_{HC}$$

3- The volume of the submerged part of the ball is equal to the volume of the water V_{water} displaced. At equilibrium, the Archimedes thrust π is equal to the weight of the volume of water displaced: $\pi = P_{HC}$

$$\rho_{water} V_{water} . g = P_{HC}$$

$$V_{water} = \frac{P_{HC}}{\rho_{water} . g} = \frac{287.79}{1000 \times 9.81} = 2.93 \times 10^{-2} m^3$$

1- Knowing that the volume of the sphere is:

$$V_S = \frac{4}{3} \times \pi \times (0.2)^3 = 3.35 \times 10^{-2} m^3$$

The fraction of the submerged volume of the ball compared to its volume:

$$\frac{V_{water}}{V_S} = \frac{2.93 \times 10^{-2} m^3}{3.35 \times 10^{-2} m^3} = 0.87 = 87\%$$

Exercise 5.3:

A very fine capillary tube with a radius r is introduced into a tank filled with water.

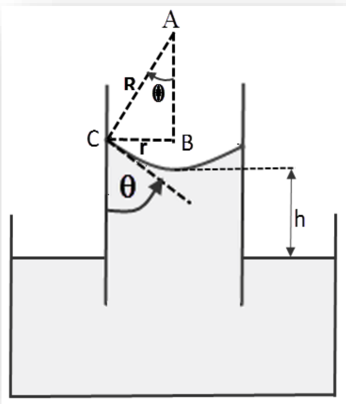
1. What phenomenon do we observe? Explain the phenomenon.

2. Demonstrate Jurin's law; the height h as a function of surface tension γ, contact angle θ, radius r of the capillary tube, density of water ρ and acceleration of gravity g.

3. Assuming that the raw sap is perfectly wetting and has the same properties as water: and, calculate the height of ascent: $\rho = 1000\ Kg.m^{-3}\ and\ \gamma = 73.10^{-3} N/m$. calculate the height of sap rise in rayon xylene channels $r = 25\mu m$

Solution

1. We observe the rise of water in the capillary tube of an height h due to the Laplace pressure difference. The reverse pressure due to the weight of the riser in the capillary will limit the rise of the water to a height h



2- The pressure difference ΔP Laplace due to surface tension γ is expressed by:

$$\Delta P = \frac{2\gamma}{R}$$

R : is the radius of curvature of the meniscus (interface between water and air).

In the triangle ABC the cosine of the contact angle θ verifies $\cos\theta = \frac{r}{R}$ hence $R = \frac{r}{\cos\theta}$

r is the radius of the capillary tube. ΔP is then written:

$$\Delta P = \frac{2\gamma\ \cos\theta}{r}$$

The water rises to a height h until the hydrostatic pressure π of the water riser in the balance tube ΔP, knowing that $\pi = \rho.g.h$, at pressure equilibrium we have: $\pi = \Delta P \Rightarrow \rho.g.h = \frac{2\gamma\ \cos\theta}{r}$

hence Jurin's law: $h = \frac{2\gamma\ \cos\theta}{r.\rho.g}$

3. The sap is perfectly wet: $\theta^0 = 0$ $r = 25\mu m = 25.10^{-6}m$

$$h = \frac{2 \times\ 73.10^{-3} \times \cos 0}{25.10^{-6} \times 1000 \times 9.81} = 0.595m$$

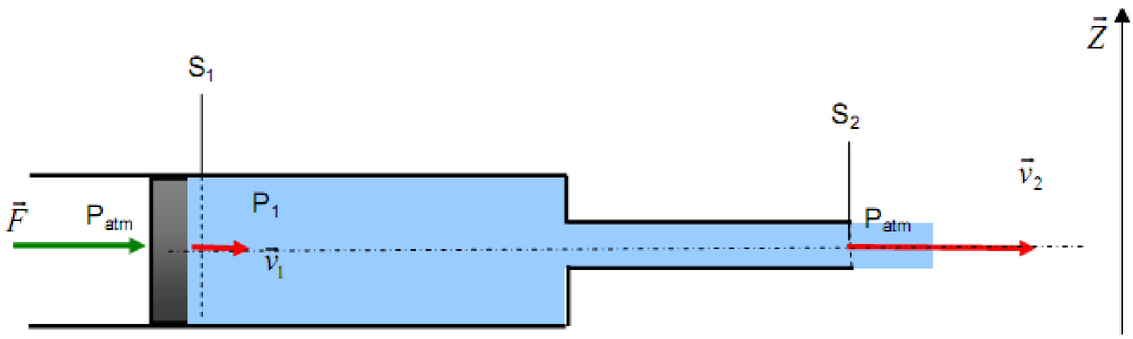
Tutorial N°6: Exercises on Bernoulli's law (hydrodynamics)

Exercise 6.1:

The figure below shows a piston that moves without friction in a cylinder of section S_1 and diameter $d_1 = 4\ cm$ filled with a perfect fluid of density$\rho = 1000\ kg/m^3$. A force F with an intensity of 62.84 N acts on the piston, at a constant speed V_1 .The fluid can escape to the outside through a cylinder of section S_2 and diameter $d_2 = 1\ cm$ at a speed V_2 and a pressure $P_2 = P_{atm} = 1\ bar$.

1- By applying the Fundamental Principle of Dynamics to the piston, determine the pressure P_1 of the fluid at section S_1 as a function of F, P_{atm} and d?
2- Write the continuity equation and determine the expression of the speed V_1 as a function of V_2?
3- By applying the Bernoulli equation, determine the flow speed V_2 as a function of P_1, P_{atm} and ρ?

   We assume that the cylinders are in a horizontal position ($Z_1 = Z_2$)



Solution

1- Applying the Fundamental Principle of Dynamics, we obtain:

$$P_1 = \frac{4F}{\pi d_1^2} + P_{atm}=1.5\text{bar}$$

2- The continuity equation:

$\pi {d_1}^2 V_1 = \pi {d_2}^2 V_2$ and $d_1 = 4d_2$

$$\pi 16 {d_2}^2 V_1 = \pi {d_2}^2 V_2 \Rightarrow V_1 = \frac{1}{16} V_2$$

3- By applying the Bernoulli equation

$$\frac{P_1}{\rho} + \frac{v_1^2}{2} = \frac{P_{atm}}{\rho} + \frac{v_2^2}{2} \Longrightarrow \frac{P_1}{\rho} + \frac{v_2^2}{2 \times 256} = \frac{P_{atm}}{\rho} + \frac{v_2^2}{2}$$

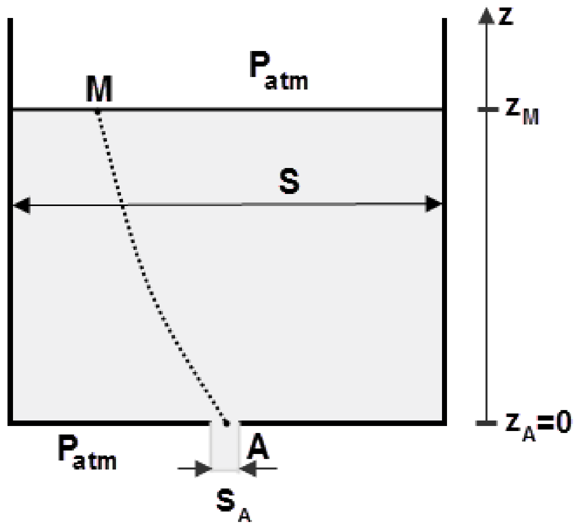
$$\frac{P_1}{\rho} - \frac{P_{atm}}{\rho} = \frac{v_2^2}{2} - \frac{v_2^2}{2 \times 256} \Longrightarrow \frac{(P_1 - P_{atm})}{\rho} = \frac{(256 - 1) \times v_2^2}{512}$$

$$\frac{(P_1 - P_{atm})}{\rho} = \frac{(255) \times v_2^2}{512}$$

$$v_2^2 = \frac{512 \times (P_1 - P_{atm})}{255 \times \rho} \Rightarrow v_2 = \sqrt{\frac{512 \times (P_1 - P_{atm})}{255 \times \rho}} = 10 m/s.$$

Exercise 6.2:

A reservoir, cubic in shape and section $S = 4m^2$ *and* $a = 2m$ .The reservoir is filled with liquid that can be emptied through an opening A pierced at its horizontal bottom and opening into the open air. A is section $S_A = 8\ cm^2$. We will assume that when it is drained, the liquid is perfect, incompressible and its flow speed is constant.



1- When emptying this reservoir, consider the streamline joining points M and A. By applying the Bernoulli relation between these two points, give the expression for the flow speed v_A liquid to the point A depending on the acceleration of gravity g and the altitude Z_M of the point M.
2- Give the relation of the volume flowQ_v at the orifice A as a function of g, Z_M and M.
3- Establish the relationship of the flow speed v_M at point M according to g, Z_M, S_A and S
4- Calculate the time necessary for the total emptying of this reservoir.

Solution

1- The Bernoulli relation between the two points M and A is written:

$$z_A + \frac{P_A}{\rho g} + \frac{v_A^2}{2g} = z_M + \frac{P_M}{\rho g} + \frac{v_M^2}{2g}$$

$$z_A = 0$$

The points M and A and being in direct contact with the air, their pressures are equal to atmospheric pressure: $P_A = P_M = P_{atm}$

$$\frac{P_{atm}}{\rho g} + \frac{v_A^2}{2g} = z_M + \frac{P_{atm}}{\rho g} + \frac{v_M^2}{2g} \Rightarrow \ v_A^2 = v_M^2 + 2.g.Z_M$$

For a perfect liquid the volume flow Q_v is constant:$S_A.v_A = S.v_M$

$$v_M = \frac{S_A.v_A}{S}$$

$S_A = 8 \times 10^{-4}\, m^2$
$S = 4\, m^2$

$$v_M = \frac{8 \times 10^{-4}\,.v_A}{4} \Rightarrow v_M = 2 \times 10^{-4}\,.v_A$$

$v_M^2 = 4 \times 10^{-8}\,.v_A^2 \ll v_A^2, we\ then\ neglect\ v_M^2\ compared\ to\ v_A^2$
$v_A^2 = 2.g.Z_M \Rightarrow\ v_A = \sqrt{2.g.Z_M}$

2- $Q_v\ = S_A.v_A = S_A.\sqrt{2.g.Z_M}$

3- $v_M = \frac{S_A.\sqrt{2.g.Z_M}}{S}$

4- Speed v_M is expressed by:

$$v_M = -\frac{dZ_M}{dt}$$

$$-\frac{dZ_M}{dt} = \frac{S_A.\sqrt{2.g.Z_M}}{S}$$

$$\frac{dZ_M}{\sqrt{Z_M}} = -\frac{S_A.\sqrt{2.g}}{S}.dt$$

We integrate from $t = 0$ until the moment T when the reservoir has been completely emptied: $Z_M = Z_A = 0\ m$

$$\int_{Z_M}^{0}\frac{dZ_M}{\sqrt{Z_M}} = -\frac{S_A.\sqrt{2.g}}{S}.\int_0^{\tau}dt$$

$$-2\sqrt{Z_M} = \frac{S_A.\sqrt{2.g}}{S}.T$$

$$-2\sqrt{Z_M} = \frac{S_A.\sqrt{2.g}}{S}.T$$

$$T = \frac{2\sqrt{Z_M}.S}{S_A.\sqrt{2.g}} = \frac{2\sqrt{2}\times 4}{8\times 10^{-4}\times\sqrt{2\times 9.81}} \approx 3193\text{ s}$$

Exercise 6.3:

The aorta is the largest artery in the body. It receives the blood that leaves the heart and distributes it to the arteries throughout the body. The heart rate for an adult is 80 beats per minute. With each beat, the heart injects a volume $v_b = 0.075\ L$ into the aorta.

1- Calculate total volume V_t of blood flowing through the aorta in one minute. Deduce the volume flow Q_v.
2- Calculate average speed v_{moy} of blood flow knowing that the diameter of the aorta is $d = 2cm.$
3- Calculate the Reynolds number R_e for flow in the aorta knowing that the dynamic viscosity of the blood is$\eta = 5 \times 10^{-3} Pa.s$ and its density is $\rho = 1060\ Kg.m^{-3}$. Deduce the flow regime.

4- Determine the critical speed v_{critical} at which the regime becomes turbulent.

5- The blood distributed by the aorta ultimately reaches the capillaries. A blood capillary is an extremely thin blood vessel of medium radius $r_c = 5\mu m$. The blood circulates there at an average speed $v_{cap} = 0.06\ cm\ s^{-1}$ Calculate the volume flow rate Q_{cap} of blood in this capillary.

6- Determine the average number N_{cap} of capillaries present in the body in a human being.

Solution

1- The total volume of blood flowing through the aorta in one minute is:

$$V_t = 80 \times v_b = 80 \times 0.075 = 6\ L$$

$$Q_v = 6L.minute^{-1} = \frac{6 \times 10^{-3}m^3}{60\ s} = 10^{-4}m^3.s^{-1}$$

2- $Q_v = v_{moy}.S$

S is the section of the aorta:

$$S = \pi.\left(\frac{d}{2}\right)^2 = \pi.\left(\frac{2 \times 10^{-2}}{2}\right)^2 = 3.14 \times 10^{-2}m^2$$

$$v_{moy} = \frac{Q_v}{S} = \frac{10^{-4}}{3.14 \times 10^{-2}} = 0.32\ m.s^{-1}$$

3- The Reynolds number is defined by:

$$R_e = \frac{\rho.v_m.d}{\eta} = \frac{1060 \times 0.32 \times 2 \times 10^{-2}}{5 \times 10^{-3}} = 1356.8$$

$$R_e < 2000\ The\ flow\ regime\ is\ laminar$$

4- The regime becomes turbulent for $R_e > 3000$. The critical speed from which the flow becomes turbulent is for $R_e = 3000$.

$$v_{\text{critical}} = \frac{\eta.R_e}{\rho.d} = \frac{5 \times 10^{-3} \times 3000}{1060 \times 2 \times 10^{-2}} = 0.71 m/s$$

5- $Q_{cap} = v_{cap}.S_{cap} = v_{cap}.\pi.r^2 = 6 \times 10^{-4} \times \pi \times (5 \times 10^{-6})^2 = 4.7 \times 10^{-14}m^3/s.$

6- $Q_v = N_{cap}.Q_{cap}$

$$N_{cap} = \frac{Q_v}{Q_{cap}} = \frac{10^{-4}}{4.7 \times 10^{-14}} = 2.13 \times 10^{10}$$