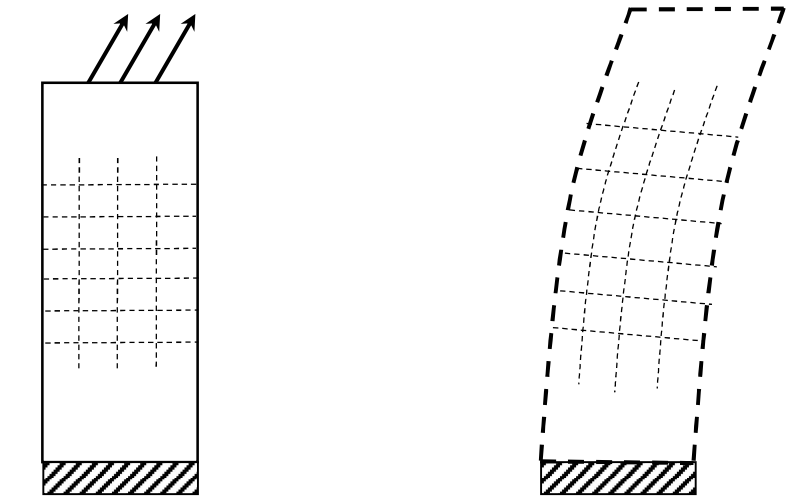
Chapter II

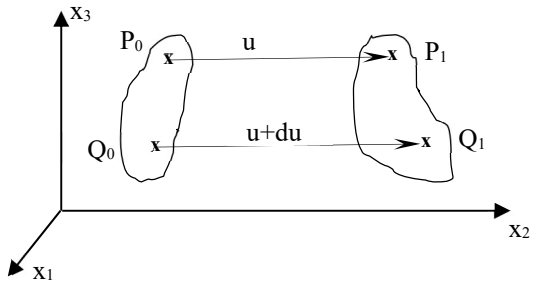
Strain Tensor

II-1- Displacement vector



Deformation of an elastic solid

In the case of linear elasticity, the assumption of small deformations is considered. Thus, we identify two neighbouring material points P_o and Q_o.



Let $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ be the displacement vector of point P

$u + du = \begin{pmatrix} u_1 + du_1 \\ u_2 + du_2 \\ u_3 + du_3 \end{pmatrix}$ the displacement vector

of point Q_o neighbouring P_o.

$$Q_0Q_1 = \begin{cases} u_1 + du_1 = u_1 + \dfrac{\partial u_1}{\partial x_1}dx_1 + \dfrac{\partial u_1}{\partial x_2}dx_2 + \dfrac{\partial u_1}{\partial x_3}dx_3 \\ u_2 + du_2 = u_2 + \dfrac{\partial u_2}{\partial x_1}dx_1 + \dfrac{\partial u_2}{\partial x_2}dx_2 + \dfrac{\partial u_2}{\partial x_3}dx_3 \\ u_3 + du_3 = u_3 + \dfrac{\partial u_3}{\partial x_1}dx_1 + \dfrac{\partial u_3}{\partial x_2}dx_2 + \dfrac{\partial u_3}{\partial x_3}dx_3 \end{cases}$$

$$Q_0Q_1 = \begin{pmatrix} u_1 + du_1 \\ u_2 + du_2 \\ u_3 + du_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$$

Q_oQ_1=u+du = u + grad(u)·dx

$Q_0Q_1 = P_0P_1 + grad(P_0P_1)\cdot(P_0Q_0)$

grad(u) : displacement gradient tensor

II-2- Strain tensor

$$grad(P_0P_1) = grad\vec{u} = \mathscr{E} + \Omega \quad \text{avec} \quad \begin{cases} \mathscr{E} = \text{Sym}(grad\vec{u}) = \frac{1}{2}(grad\vec{u} + grad^t\vec{u}) \\ \Omega = \text{Antisym}(grad\vec{u}) = \frac{1}{2}(grad\vec{u} - grad^t\vec{u}) \end{cases}$$

$Q_0Q_1 = P_0P_1 + \mathscr{E}\cdot(P_0Q_0) + \Omega\cdot(P_0Q_0)$ or u+du = u + $\mathscr{E}$·dx + Ω·dx

Consequently, an elementary domain surrounding a point P undergoes the sum of three transformations:

- a translation of direction vector $P_0P_1 = u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$
- a rotation around point P_0 by an angle ω dual vector of an antisymmetric second-order tensor Ω (rotation tensor) such that :

$$\Omega = \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) & 0 \end{pmatrix} \quad \omega = \begin{pmatrix} \omega_{23} \\ \omega_{13} \\ \omega_{12} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) \end{pmatrix}$$

$$\Omega = (*\,\omega) = \begin{pmatrix} 0 & -\omega_{12} & \omega_{13} \\ \omega_{12} & 0 & -\omega_{23} \\ -\omega_{13} & \omega_{23} & 0 \end{pmatrix} \quad \text{with} \quad \omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

- pure strain (deformation) defined by the following tensor :

- $\mathscr{E}=\begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$

$$\mathscr{E}=\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix} \qquad \text{with} \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$$

Case of cylindrical coordinates:

$$\begin{aligned} x_1 &= \mathrm{r}\cos(\theta) \\ x_2 &= \mathrm{r}\sin(\theta) \\ x_3 &= x_3 \end{aligned} \qquad \text{et} \qquad u = \begin{cases} u_1 = u_1(\mathrm{r},\theta,\mathrm{x}_3) \\ u_2 = u_2(\mathrm{r},\theta,\mathrm{x}_3) \\ u_3 = u_3(\mathrm{r},\theta,\mathrm{x}_3) \end{cases}$$

$$grad\vec{u} = \begin{pmatrix} \frac{\partial u_1}{\partial r} & \frac{1}{r}(\frac{\partial u_1}{\partial \theta}-u_2) & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial r} & \frac{1}{r}(\frac{\partial u_2}{\partial \theta}+u_1) & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial r} & \frac{1}{r}\frac{\partial u_3}{\partial \theta} & \frac{\partial u_3}{\partial x_3} \end{pmatrix} \qquad \mathscr{E}=\begin{pmatrix} \frac{\partial u_1}{\partial r} & \frac{1}{2}(\frac{\partial u_2}{\partial r}+\frac{1}{r}\frac{\partial u_1}{\partial \theta}-\frac{u_2}{r}) & \frac{1}{2}(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial r}) \\ \varepsilon_{r\theta} & \frac{1}{r}(\frac{\partial u_2}{\partial \theta}+u_1) & \frac{1}{2}(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial \theta}) \\ \varepsilon_{r3} & \varepsilon_{\theta 3} & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

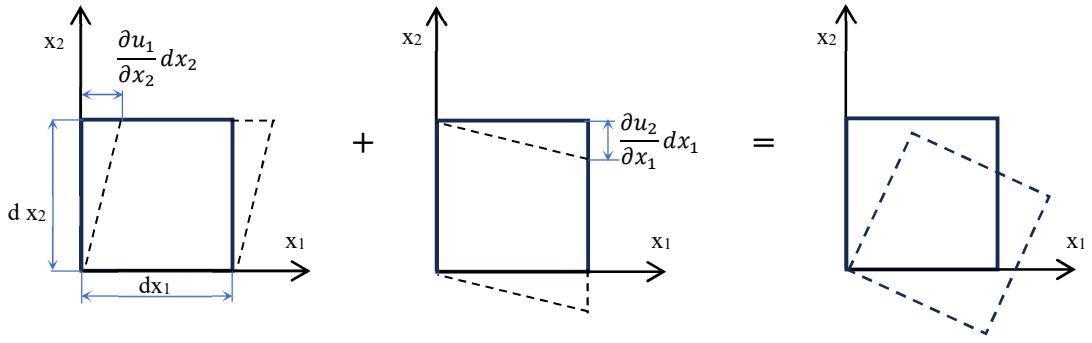
$$\Omega = \begin{pmatrix} 0 & \frac{1}{2}(\frac{1}{r}\frac{\partial u_1}{\partial \theta}-\frac{u_2}{r}-\frac{\partial u_2}{\partial r}) & \frac{1}{2}(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial r}) \\ \frac{1}{2}(\frac{\partial u_2}{\partial r}-\frac{1}{r}\frac{\partial u_1}{\partial \theta}+\frac{u_2}{r}) & 0 & \frac{1}{2}(\frac{\partial u_2}{\partial x_3}-\frac{\partial u_3}{\partial \theta}) \\ \frac{1}{2}(\frac{\partial u_3}{\partial r}-\frac{\partial u_1}{\partial x_3}) & \frac{1}{2}(\frac{\partial u_3}{\partial \theta}-\frac{\partial u_2}{\partial x_3}) & 0 \end{pmatrix}$$

Two-dimensional case study :

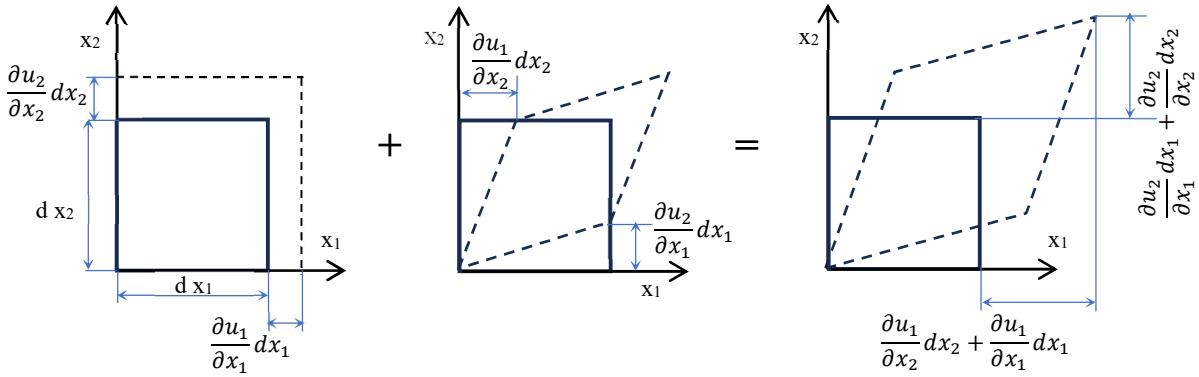
we consider the case of the two-dimensional deformation of a rectangular element of original dimensions dx_1 by dx_2.

$$\Omega = \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & 0 \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathscr{E} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & 0 \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

if $\varepsilon_{ij}=0 \Rightarrow \frac{\partial u_1}{\partial x_2} = -\frac{\partial u_2}{\partial x_1} \Rightarrow \Omega = \begin{pmatrix} 0 & \frac{\partial u_1}{\partial x_2} & 0 \\ \frac{\partial u_2}{\partial x_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial u_1}{\partial x_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \frac{\partial u_2}{\partial x_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$



if $\omega_{ij}=0 \Rightarrow \frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1} \Rightarrow \mathscr{E} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & 0 \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial u_2}{\partial x_2} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{\partial u_1}{\partial x_2} & 0 \\ \frac{\partial u_2}{\partial x_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

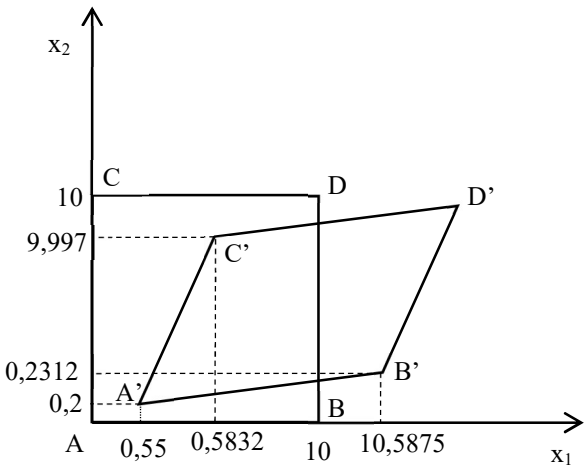


Example :

A square ABCD with side length 10 is drawn at point A in the plane (A/ x_1, x_2). After loading, and with respect to a fixed coordinate system, we obtain the parallelogram A'B'C'D'.

Determine the components ε_{ij} of the matrix associated with the pure strain tensor and the rotation of the medium.

Solution :



The problem being plan ⇒ $\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$

$\Omega_{13} = \Omega_{23} = 0$

$$\mathscr{E} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Omega = \begin{pmatrix} 0 & -\omega_{12} & 0 \\ \omega_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = \frac{(BB')_{x_1} - (AA')_{x_1}}{AB}$$

$$= \frac{(10.5875 - 10) - (0.55 - 0)}{10} = 3.75 \cdot 10^{-3}$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = \frac{(CC')_{x_2} - (AA')_{x_2}}{AC} = \frac{(9.997 - 10) - (0.2 - 0)}{10} = -20.3 \cdot 10^{-3}$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\left(\frac{(CC')_{x_1} - (AA')_{x_1}}{AC} + \frac{(BB')_{x_2} - (AA')_{x_2}}{AB}\right)$$

$$= \frac{1}{2}\left(\frac{0.5832 - 0.55}{10} + \frac{0.2312 - 0.2}{10}\right) = \frac{1}{2}(3.32 \cdot 10^{-3} + 3.12 \cdot 10^{-3}) = 3.22 \cdot 10^{-3}$$

$$\Omega_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\left(\frac{(CC')_{x_1} - (AA')_{x_1}}{AC} - \frac{(BB')_{x_2} - (AA')_{x_2}}{AB}\right)$$

$$= \frac{1}{2}\left(\frac{0.5832 - 0.55}{10} - \frac{0.2312 - 0.2}{10}\right) = \frac{1}{2}(3.32 \cdot 10^{-3} - 3.12 \cdot 10^{-3}) = 0.1 \cdot 10^{-3}$$

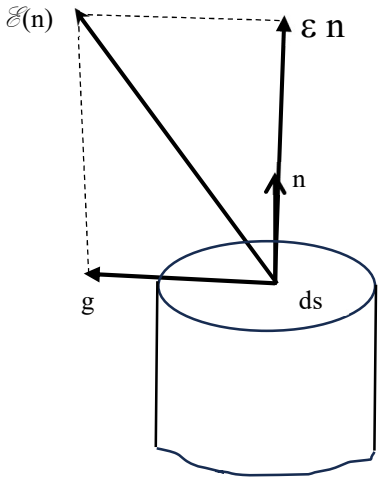
Le tenseur des déformations pures sera :

$$\mathscr{E} = \begin{pmatrix} 3.75 \cdot 10^{-3} & 3.22 \cdot 10^{-3} & 0 \\ 3.22 \cdot 10^{-3} & -20.3 \cdot 10^{-3} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Le tenseur de rotation et le vecteur de rotation sont respectivement :

$$\Omega = \begin{pmatrix} 0 & -0.1 \cdot 10^{-3} & 0 \\ 0.1 \cdot 10^{-3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{et} \qquad \omega = \begin{pmatrix} 0 \\ 0 \\ 0.1 \cdot 10^{-3} \end{pmatrix}$$

II-3- decomposition of the deformation



$\vec{n}$ vector normal to the surface ds

The strain vector is defined by : :

$\mathscr{E}(\vec{n}) = \varepsilon \,.\, \vec{n} + \vec{g}$ is composed of:

Normal deformation (elongation) along the vector n :

$$\varepsilon = \vec{n}^t \, \mathscr{E} \, \vec{n}$$

Shear deformation (slip) on the plane of normal n :

$$g = |(* \vec{n}) \mathscr{E} \vec{n}| = \sqrt{|\mathscr{E}(\vec{n})|^2 - \varepsilon^2}$$

II-4- Principal Strains and principal directions

The tensor $\mathscr{E}$ being symmetrical, therefore there exists a basis $\{X_1, X_2, X_3\}$ called principal basis on which the matrix ($\mathscr{E}$) is diagonal:

$$\mathscr{E} = \begin{pmatrix} \varepsilon_I & 0 & 0 \\ 0 & \varepsilon_{II} & 0 \\ 0 & 0 & \varepsilon_{III} \end{pmatrix}$$

The principal strains (or elongations) ε_I, ε_{II} and ε_{III} represent the eigenvalues of the matrix ($\mathscr{E}$) which can be obtained by solving the following equation:

$$\det(\mathscr{E} - \lambda \mathrm{I}) = 0$$

Expanding the determinant produces a cubic equation in terms of λ *(called characteristic equation)*:

$$det[\mathscr{E} - \lambda I] = -\lambda^3 + I_a \lambda^2 - I_B \lambda + I_C = 0$$

The roots of the characteristic equation determine the allowable values for λ $(\lambda_1, \lambda_2, \lambda_3)$ such as :

$\varepsilon_I = \lambda_1$ $\varepsilon_{II} = \lambda_2$ $\varepsilon_{III} = \lambda_3$

Where the scalars I_a, I_b, and I_c are called the *fundamental invariants of the tensor* $\mathscr{E}$ such as :

$$I_a = tr(\mathscr{E}) = \varepsilon_{ii} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \lambda_1 + \lambda_2 + \lambda_3$$

$$I_b = \tfrac{1}{2}(\varepsilon_{ii}\varepsilon_{jj} - \varepsilon_{ij}\varepsilon_{ij}) = \begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{vmatrix} + \begin{vmatrix} \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{23} & \varepsilon_{33} \end{vmatrix} + \begin{vmatrix} \varepsilon_{11} & \varepsilon_{13} \\ \varepsilon_{13} & \varepsilon_{33} \end{vmatrix} =$$

$$= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$$

$$I_c = \det(\mathscr{E}) = det[\varepsilon_{ij}] = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$$

Each of the values of λ (or ε_i) can be substituted into the following vector relation to find the principal directions X_1 , X_2 et X_3 :

$$(\mathscr{E} - \varepsilon_i I)\,\vec{X}_i = \vec{0}$$

The unit vectors X_1, X_2 and X_3 of the principal basis verify the following vector relations:

$$\begin{cases} X_1 = X_2 \wedge X_3 \\ X_2 = X_3 \wedge X_1 \\ X_3 = X_1 \wedge X_2 \end{cases}$$

III-5- Spherical and Deviatoric Strains

In particular applications it is convenient to decompose the strain tensor into two parts called spherical and deviatoric strain tensors.

$$\mathscr{E} = \mathscr{E}_s + \mathscr{E}_d$$

The spherical strain tensor is defined by

$$\mathscr{E}_s = \frac{I_a}{3} I \qquad (I_a = \mathrm{tr}(\mathscr{E}))$$

The deviatoric strain tensor is defined by

$$\mathscr{E}_d = \mathscr{E} - \mathscr{E}_s = \mathscr{E} - \frac{I_a}{3} I$$

The spherical strain represents only volumetric deformation ($\mathrm{tr}(\mathscr{E}_s) = \mathrm{tr}(\mathscr{E})$) and is an isotropic tensor, being the same in all coordinate systems. The deviatoric strain tensor then accounts for changes in shape of material elements. It can be shown that the principal directions of the deviatoric strain are the same as those of the strain tensor. The deviatoric tensor is responsible for shear strain ($\mathrm{tr}(\mathscr{E}_d) = 0$).

Example:

We consider the following displacement field $u = P_0 P_1 = (u_1, u_2, u_3)^T$:

$$u_1 = 730.10^{-6}\, x_1 + 350.10^{-6}\, x_2$$
$$u_2 = 430.10^{-6}\, x_1 + 145.10^{-6}\, x_2$$
$$u_3 = -375.10^{-6}\, x_3$$

1°) Determine the Grad(u) tensor. Deduce the pure strain and rotation tensors, as well as the rotation vector.

2°) Determine the principal strains and principal directions

3°) Determine the angle (x_1, X_1) that the x_1 (or e_1) makes with the principal axis X_1

4°) Determine the unit elongation and the relative slip in the direction n making an angle of 45° with the x_1 axis (or e_1) and perpendicular to the x_3 axis (or e_3).

5°) decompose the strain tensor into a spherical part and a deviatoric part.

Solution :

1°) $$grad(U) = \begin{bmatrix} \partial u/\partial x_1 & \partial u/\partial x_2 & \partial u/\partial x_3 \\ \partial v/\partial x_1 & \partial v/\partial x_2 & \partial v/\partial x_3 \\ \partial w/\partial x_1 & \partial w/\partial x_2 & \partial w/\partial x_3 \end{bmatrix} = \begin{pmatrix} 730 & 350 & 0 \\ 430 & 145 & 0 \\ 0 & 0 & -375 \end{pmatrix} \cdot 10^{-6}$$

$$\mathscr{E} = sym(grad(U)) = \frac{1}{2}(grad(U) + grad^t(U)) = \begin{pmatrix} 730 & 390 & 0 \\ 390 & 145 & 0 \\ 0 & 0 & -375 \end{pmatrix} \cdot 10^{-6}$$

$$\Omega = antisym(grad(U)) = \frac{1}{2}(grad(U) - grad^t(U)) = \begin{pmatrix} 0 & -40 & 0 \\ 40 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot 10^{-6}$$

$$\omega = \begin{pmatrix} 0 \\ 0 \\ 40 \end{pmatrix} \cdot 10^{-6}$$

2°) Let ε_I , ε_{II} and ε_{III} be the principal elongations of $\mathscr{E}$ and X_1 , X_2 et X_3 the corresponding principal directions.

$$det(\mathscr{E} - \lambda I) = 0 \Rightarrow \left| \begin{pmatrix} 730 & 390 & 0 \\ 390 & 145 & 0 \\ 0 & 0 & -375 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = \begin{vmatrix} 730-\lambda & 390 & 0 \\ 390 & 145-\lambda & 0 \\ 0 & 0 & -375-\lambda \end{vmatrix} = 0$$

$(-375-\lambda)[(730-\lambda)(145-\lambda)-390^2]=0 \quad \Rightarrow \quad (-375-\lambda)(\lambda^2-875\lambda-46250) = 0$

$$\Rightarrow \begin{cases} \lambda_1 = -375 \\ \lambda_2 = -50 \text{ et } \lambda_3 = 925 \end{cases}$$

$\varepsilon_I > \varepsilon_{II} > \varepsilon_{III} \quad \Rightarrow \quad \varepsilon_I = 925\ 10^{-6}$; $\varepsilon_{II} = -50\ 10^{-6}$; $\varepsilon_{III} = -375\ 10^{-6}$

Principal directions.:

For $\varepsilon_I = 925\ 10^{-6}$

$$(\mathscr{E} - \varepsilon_I I)\, X_1 = 0 \Rightarrow \begin{pmatrix} 730-925 & 390 & 0 \\ 390 & 145-925 & 0 \\ 0 & 0 & -375-925 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -195\, a_1 + 390\, b_1 = 0 \\ 390\, a_1 - 780\, b_1 = 0 \\ -1300\, c_1 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = \frac{390}{195} b_2 = 2b_1 \\ \Rightarrow \quad c_1 = 0 \end{cases}$$

X_1 unit vector $\Rightarrow$ $\quad a_1^2 + b_1^2 + c_1^2 = 1 \quad \Rightarrow \quad 4\,b_1^2 + b_1^2 + 0 = 1 \Rightarrow b_1 = \frac{1}{\sqrt{5}} \Rightarrow a_1 = \frac{2}{\sqrt{5}}$

For $\varepsilon_{II} = -50\ 10^{-6}$

$(\mathscr{E} - \varepsilon_{II}I)\ X_2 = 0 \quad \Rightarrow \begin{pmatrix} 370+50 & 390 & 0 \\ 390 & 145+50 & 0 \\ 0 & 0 & -375+50 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{cases} 780\,a_2 + 390\,b_2 = 0 \\ 390\,a_2 + 195\,b_2 = 0 \\ -325\,c_2 = 0 \end{cases} \Rightarrow \begin{cases} b_2 = \frac{390}{195} b_2 = 2\text{-}a_2 \\ \Rightarrow \quad c_2 = 0 \end{cases}$$

X_2 unit vector $\Rightarrow$ $\quad a_2^2 + b_2^2 + c_2^2 = 1 \quad \Rightarrow \quad a_2^2 + 4a_2^2 + 0 = 1 \qquad a_2 = \frac{-1}{\sqrt{5}} \Rightarrow a_2 = \frac{2}{\sqrt{5}}$

For $\varepsilon_{III} = -375\ 10^{-6}$

$(\mathscr{E} - \varepsilon_{III}I)\ X_3 = 0 \quad \Rightarrow \begin{pmatrix} 730+375 & 390 & 0 \\ 390 & 145+375 & 0 \\ 0 & 0 & -375+375 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left.\begin{matrix} 1105a_3 + 390b_3 = 0 \\ 390a_3 + 520b_3 = 0 \\ 0 \cdot c_3 = 0 \end{matrix}\right\} \Rightarrow \ a_3 = b_3 = 0 \ \ (\det \neq 0)$$

the value of c_3 is arbitrary

X_3 unit vector $\Rightarrow$ $\quad a_3^2 + b_3^2 + c_3^2 = 1 \quad c_3^2 = 1$ we take $c_3 = +1$

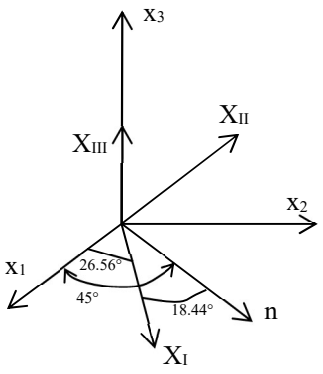
$$X_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad ; \quad X_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \quad ; \quad X_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

3°) $tg(x_1, X_1) = \frac{1/\sqrt{5}}{2/\sqrt{5}} = \frac{1}{2} \quad \Rightarrow \quad \theta = (x_1, X_1) = 26.56°$

4°) $(x_1\ ,\ n) = 45° \quad \Rightarrow \quad n = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix}$

$$\varepsilon_n = n^t E\, n = \left(\frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \quad 0\right) \begin{pmatrix} 730 & 390 & 0 \\ 390 & 145 & 0 \\ 0 & 0 & -375 \end{pmatrix} \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix} \cdot 10^{-6}$$

$$= \left(\frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \quad 0\right) \begin{pmatrix} 1120 \\ 535 \\ 0 \end{pmatrix} \frac{\sqrt{2}}{2} \cdot 10^{-6} = 827.5 \cdot 10^{-6}$$



$$g_n = \sqrt{|\mathscr{E}n|^2 - \varepsilon^2} = \left(\begin{vmatrix} 1120 \\ 535 \\ 0 \end{vmatrix}^2 - 827.5^2 \right)^{1/2} \cdot 10^{-6} = (1120^2 + 535^2 - 827.5^2)^{1/2} \cdot 10^{-6}$$

$$= 292.5 \cdot 10^{-6}$$

5°) $\mathscr{E} = \mathscr{E}_s + \mathscr{E}_d$

the spherical part :

$\mathscr{E}_s = \frac{I_a}{3} \mathrm{I}$ with $I_a = \mathrm{tr}\,(\mathscr{E}) = (925 - 50 - 375)10^6$

$$= (730 + 145 - 375)10^{-6} = 500 \cdot 10^{-6}$$

$$\Rightarrow \quad \mathscr{E}_s = \frac{1}{3}\begin{pmatrix} 500 & 0 & 0 \\ 0 & 500 & 0 \\ 0 & 0 & 500 \end{pmatrix} 10^{-6} = \begin{pmatrix} 166.67 & 0 & 0 \\ 0 & 166.67 & 0 \\ 0 & 0 & 166.67 \end{pmatrix} 10^{-6}$$

the deviatoric part :

$$\mathscr{E}_d = \mathscr{E} - \frac{I_a}{3} I \quad \mathscr{E}_d = \begin{pmatrix} 730 & 390 & 0 \\ 390 & 145 & 0 \\ 0 & 0 & -375 \end{pmatrix} 10^{-6} - \frac{1}{3}\begin{pmatrix} 500 & 0 & 0 \\ 0 & 500 & 0 \\ 0 & 0 & 500 \end{pmatrix} 10^{-6}$$

$$\Rightarrow \mathscr{E}_d = \frac{1}{3}\begin{pmatrix} 1690 & 1170 & 0 \\ 1170 & -65 & 2 \\ 0 & 2 & -1625 \end{pmatrix} 10^{-6} = \begin{pmatrix} 563.33 & 390 & 0 \\ 390 & -21.67 & 2 \\ 0 & 2 & -541.67 \end{pmatrix} 10^{-6}$$