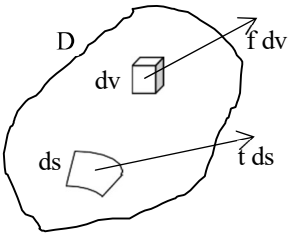
Chapter IV

Stress Tensor

IV-1 Types of external forces

Let D be a material domain of volume (V) and exterior surface (S).



The external forces applied to the domain are of two types:

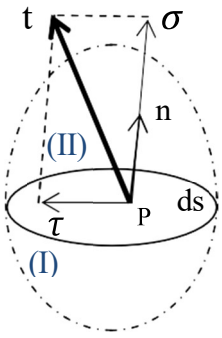
- Body forces acted on all particles of D.

On a volume dv, the resultant force is f·dv (f: [N/m³])

- Surface forces exerted on the particles constituting the boundary(S) of D.

On a surface ds, the resultant force is t·ds (t: [N/m²])

IV-2- Stress Vector



Consider an elementary domain diIVded into two regions (I) and (II). The two regions are separated by a surface (ds).

The resultant contact forces of (II) on (I) are: $\vec{t}\cdot ds$

$(\vec{t})$ is the **stress vector** at point P relative to $\vec{n}$.

$$\vec{t} = \vec{\sigma} + \vec{\tau}$$

σ is the projection of t onto $\vec{n}$, called the **normal stress**.

τ is the projection of t onto (ds), called the **shear stress**.

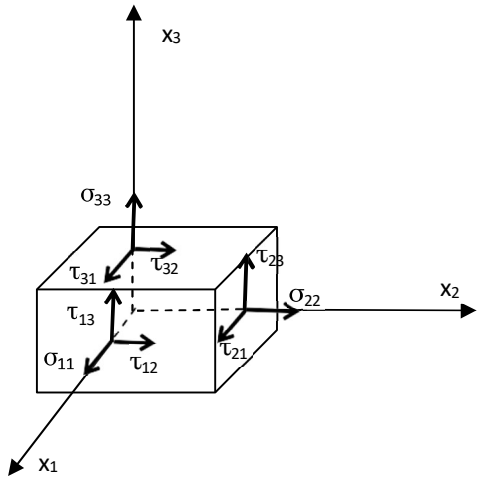
IV-3- Stress Tensor

Cauchy's Theorem: At every point and at every instant, the dependence of the stress vector t on the normal n is linear. Therefore, there exists a second-order tensile field such that:

$$\vec{t} = \Sigma\,\vec{n} \qquad \text{ou} \quad t_i = \sigma_{ij}\, n_j$$

The stress tensor is defined by:

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{pmatrix}$$



σ_{ii} : normal stress

τ_{ij} : shear stress

Thus, the stress vector can therefore be decomposed into:

normal stress $\sigma = \vec{n}^{\,t}\ \Sigma\, \vec{n}$

shear stress $\tau = |(*\,\vec{n})\ \Sigma\, \vec{n})| = \sqrt{|\Sigma(\vec{n})|^2 - \sigma^2}$

IV-4- Equilibrium equations

The equations of equilibrium express the vanishing of the sum of forces and the sum of moments about a point.

$$\iint_s \vec{t}\ ds + \iiint_v \vec{f}\ dv = \vec{0}$$

$$\iint_s \overrightarrow{OP} \wedge \vec{t}\ ds + \iiint_v \overrightarrow{OP} \wedge \vec{f}\, dv = \vec{0}$$

By projecting onto the axis e_i (i=1, 2, 3) we obtain :

$$\iint_s \sigma_{ij}\, n_j\ ds + \iiint_v f_i\ dv = 0$$

$$\iint_s (x_i\, \sigma_{jk} n_k - x_j\, \sigma_{ik} n_k)\, ds + \iiint_v (x_i f_j - x_j f_i)\, dv = 0$$

The divergence theorem allows the transformation of the double integral into a triple integral :

$$\iiint_v \left(\frac{\partial \sigma_{ij}}{\partial x_j} + f_i\right) dv = 0$$

$$\iiint_v \left(\frac{\partial (x_i\, \sigma_{jk} - x_j\, \sigma_{ik})}{\partial x_k} + x_i f_j - x_j f_i\right) dv = 0$$

The second equilibrium equation reflects the symmetry of the stress tensor : $\sigma_{ij} = \sigma_{ji}$

The stress tensor can then be written as:

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33} \end{pmatrix}$$

The first equilibrium equation reflects the local equilibrium of forces: $\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0$

or in vectorial form: $\vec{f} + \text{div}(\Sigma) = \vec{0}$

These equations are written in:

Cartesian coordinates

$$f_1+\frac{\partial \sigma_{11}}{\partial x_1}+\frac{\partial \sigma_{12}}{\partial x_2}+\frac{\partial \sigma_{13}}{\partial x_3}=0$$
$$f_2+\frac{\partial \sigma_{12}}{\partial x_1}+\frac{\partial \sigma_{22}}{\partial x_2}+\frac{\partial \sigma_{23}}{\partial x_3}=0$$
$$f_3+\frac{\partial \sigma_{13}}{\partial x_1}+\frac{\partial \sigma_{23}}{\partial x_2}+\frac{\partial \sigma_{33}}{\partial x_3}=0$$

cylindrical coordinates:

$$f_r+\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\left(\frac{\partial \sigma_{r\theta}}{\partial \theta}+\sigma_{rr}+\sigma_{\theta\theta}\right)+\frac{\partial \sigma_{r3}}{\partial x_3}=0$$
$$f_\theta+\frac{\partial \sigma_{r\theta}}{\partial r}+\frac{1}{r}\left(\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+2\sigma_{r\theta}\right)+\frac{\partial \sigma_{\theta 3}}{\partial x_3}=0$$
$$f_3+\frac{\partial \sigma_{r3}}{\partial r}+\frac{1}{r}\left(\frac{\partial \sigma_{\theta 3}}{\partial \theta}+\sigma_{r3}\right)+\frac{\partial \sigma_{33}}{\partial x_3}=0$$

IV-5- Principal Stress and principal directions

In the principal basis frame $\{X_1 , X_2 , X_3\}$ the stress tensor is written:

$$\Sigma = \begin{pmatrix} \sigma_I & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{pmatrix}$$

To determine the principal elongations σ_I , σ_{II} et σ_{III}, the following equation must be solved:

$$\det(\Sigma - \lambda I) = 0$$

Expanding the determinant produces a cubic equation in terms of λ *(called characteristic equation)*:

$$det[\Sigma - \lambda I] = -\lambda^3 + s_1\lambda^2 - s_2\lambda + s_3 = 0$$

The roots of the characteristic equation determine the allowable values for λ $(\lambda_1, \lambda_2, \lambda_3)$ such as :

$\sigma_I=\lambda_1$ $\sigma_{II}=\lambda_2$ $\sigma_{III}=\lambda_3$

Where the scalars s_1, s_2, and s_3 are called the fundamental invariants of the tensor Σ such as :

$$s_1 = tr(\Sigma) = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_I + \sigma_{II} + \sigma_{III}$$
$$s_2 = \tfrac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}) = \begin{vmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{vmatrix} + \begin{vmatrix}\sigma_{22} & \sigma_{23}\\ \sigma_{23} & \sigma_{33}\end{vmatrix} + \begin{vmatrix}\sigma_{11} & \sigma_{13}\\ \sigma_{13} & \sigma_{33}\end{vmatrix} =$$
$$= \sigma_I\sigma_{II} + \sigma_{II}\sigma_{III} + \sigma_I\sigma_{III}$$
$$s_3 = \det(\Sigma) = det[\sigma_{ij}] = \sigma_I \cdot \sigma_{II} \cdot \sigma_{III}$$

To determine the principal directions X_1 , X_2 et X_3 the following vectorial equation must be solved:

$$(\Sigma - \sigma_i I)\,\vec{X}_i = \vec{0}$$

The unit vectors X_1, X_2 and X_3 of the principal basis verify the following vector relations:

$$\begin{cases} X_1 = X_2 \wedge X_3 \\ X_2 = X_3 \wedge X_1 \\ X_3 = X_1 \wedge X_2 \end{cases}$$

IV-6- Geometric representation of the stress state at a point (Mohr's circles)

Given the plane of the points (σ,τ), we seek the domain generated by the points M(σ,τ) when the end of the unit vector n describes the sphere with center P and radius 1.

The main constraints are noted so as to have: $\sigma_I > \sigma_{II} > \sigma_{III}$

The region generated by the points M(σ, τ) is bounded by the three circles whose equations are as follows:

$$\left[\sigma - \frac{\sigma_I+\sigma_{II}}{2}\right]^2 + \tau^2 = \left[\frac{\sigma_I-\sigma_{II}}{2}\right]^2$$

$$\left[\sigma - \frac{\sigma_I+\sigma_{III}}{2}\right]^2 + \tau^2 = \left[\frac{\sigma_I-\sigma_{III}}{2}\right]^2$$

$$\left[\sigma - \frac{\sigma_{II}+\sigma_{III}}{2}\right]^2 + \tau^2 = \left[\frac{\sigma_{II}-\sigma_{III}}{2}\right]^2$$

then

Circle (1,2) :

$C12 = (\sigma I + \sigma II)/2$

$R12 = (\sigma I - \sigma II)/2$

Circle (1,3) :

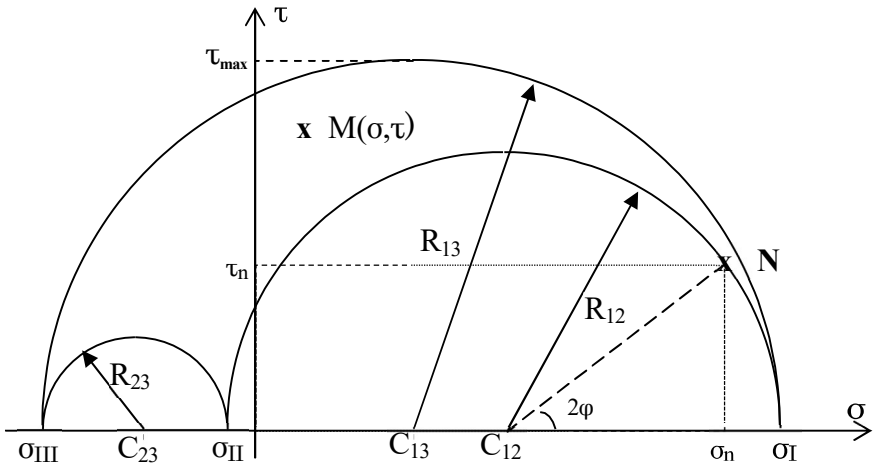
$C_{13} = (\sigma_I + \sigma_{III})/2$

$R_{13} = (\sigma_I - \sigma_{III})/2$

Circle (2,3)

$C_{23} = (\sigma_{II} + \sigma_{III})/2$

$R_{23} = (\sigma_{II} - \sigma_{III})/2$



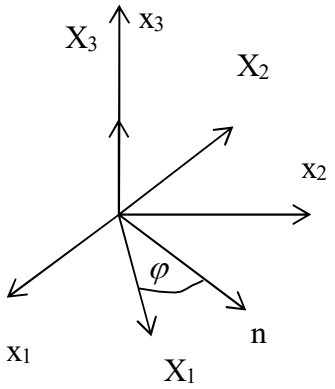
In general, for the circle (i,j) : the center $(0, C_{ij})$ with $C_{ij} = (\sigma_i + \sigma_j)/2$ et $(\sigma_i > \sigma_j)$

the radius R_{ij} with $R_{ij} = (\sigma_i - \sigma_j)/2$

Special case:

If the vector $\vec{n}$ lies in the (X_1, X_2) plane, then the set of points N(σ,τ) -figure below- will be the circle centered at C_{12} with radius R_{12} defined by the equation:

$$\left[\sigma - \frac{\sigma_I+\sigma_{II}}{2}\right]^2 + \tau^2 = \left[\frac{\sigma_I-\sigma_{II}}{2}\right]^2$$



Considering $\vec{n} = \begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\\ 0\end{pmatrix}$ with $\varphi = (\vec{n}, \vec{X}_1)$ polar angle

We can write :

$$\sigma_n = \vec{n}_n^t \, \Sigma \, \vec{n}_n \qquad \Rightarrow \qquad \sigma_n = \frac{1}{2}(\sigma_I + \sigma_{II}) + \frac{1}{2}(\sigma_I - \sigma_{II})\cos(2\varphi)$$

$$\tau_n = |(* \vec{n}_n) \, \Sigma \, \vec{n}_n)| = \sqrt{[\Sigma(\vec{n}_n)]^2 - \sigma_n^2} \qquad \Rightarrow \qquad \tau_n = \frac{1}{2}(\sigma_I - \sigma_{II})\sin(2\varphi)$$

IV-7- Spherical and Deviatoric Stress

In particular applications it is convenient to decompose the stress tensor into two parts called spherical and deviatoric parts.

$\Sigma = \Sigma_s + \Sigma_d$ with $\Sigma_s = \frac{s_1}{3} I$ ($s_1 = tr(\Sigma)$) : the spherical part of Σ

$\Sigma_d = \Sigma - \frac{s_1}{3} I$: the deviatoric part of Σ

IV-8- Octahedral tension and shear

Octahedral tension and shear are defined for the direction

$$n_o = \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} \quad \text{such as} \quad \alpha^2 = \beta^2 = \gamma^2$$

$\alpha^2 + \beta^2 + \gamma^2 = 1$ $\qquad \alpha^2 = \beta^2 = \gamma^2 = 1/3 \qquad \Rightarrow \qquad n_o = \frac{\sqrt{3}}{3}\begin{pmatrix}\pm 1\\ \pm 1\\ \pm 1\end{pmatrix}$

Octahedral tension σ_o :

$$\sigma_o = \vec{n}_o^t \, \Sigma \, \vec{n}_o = \frac{1}{3}(\sigma_I + \sigma_{II} + \sigma_{III}) = \frac{1}{3}s_1$$

Octahedral shear τ_o :

$$\tau_o = |(* \vec{n}_o) \, \Sigma \, \vec{n}_o)| = \sqrt{[\Sigma(\vec{n}_o)]^2 - \sigma_o^2} = \frac{1}{3}\sqrt{(\sigma_I - \sigma_{II})^2 + (\sigma_I - \sigma_{III})^2 + (\sigma_{II} - \sigma_{III})^2}$$

IV-9- Special states of stress

a) Spherical state of stress: cases where the three principal stresses are vanishes

$$\sigma_I = \sigma_{II} = \sigma_{III} = (tr(\Sigma))/3 = s_1/3$$

b) Uniaxial state of stress: cases where the two principal stresses are vanishes

$$\sigma_{II} = \sigma_{III} = 0 \quad \text{and} \quad \sigma_I \neq 0$$

c) Pure shear state: case where all stress components σ_{ij} are zero except for a single shear component. $\sigma_{ij} = 0$ and $\tau_{12} \neq 0$

d) Plane stress state: cases where the stresses σ_{11}, σ_{22}, and τ_{12} are independent of x_3.

$$\Sigma = \begin{pmatrix} \sigma_{11}(x_1, x_2) & \tau_{12}(x_1, x_2) & 0 \\ \tau_{12}(x_1, x_2) & \sigma_{22}(x_1, x_2) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$