Contents

1 Introduction to the mathematical modelling **1**

1.1 Mathematical modelling 1

1.1.1 What is mathematical modelling? 1

1.1.2 What objectives can modelling achieve? 2

1.1.3 Classifications of models 2

1.1.4 Model building 3

1.2 Applied mathematics 5

1.2.1 What is applied mathematics 5

List of Figures

1.1 The key components studied under the umbrella of applied mathematics: (1) mathematical science, (2) domain-specific knowledge, and (3) a few 'math-adjacent' disciplines 4

1.2 The key components studied under the umbrella of applied mathematics: (1) mathematical science, (2) domain-specific knowledge, and (3) a few 'math-adjacent' disciplines 5

Chapter 1

Introduction to the mathematical modelling

1.1 Mathematical modelling

1.1.1 What is mathematical modelling?

In general, models describe our beliefs about how the world functions. In mathematical modelling, we translate those beliefs into mathematical language. This has many advantages

1. Mathematics is a very precise language. Hence, we can formulate our ideas precisely and so are less likely to let implicit assumptions slip by.

2. Mathematics is a concise language, with well-defined rules for manipulations.

3. All the results that mathematicians have proved over hundreds of years are at our disposal.

4. Computers can be used to perform numerical calculations.

There is a large element of compromise in mathematical modelling. The majority of interacting systems in the real world are far too complicated to model in their entirety. Hence the ***first level of compromise*** is to identify the most important parts of the system. These will be included in the model; the rest will be excluded. The ***second level of compromise*** concerns the amount of mathematical manipulation which is worthwhile. Although mathematics has the potential to prove general results, these results depend critically on the form of equations used. Small changes in the structure of equations may

require enormous changes in the mathematical methods. Using computers to handle the model equations may never lead to elegant results, but it is much more robust against alterations.

1.1.2 What objectives can modelling achieve?

Mathematical modelling can be used for several different reasons. How well any particular objective is achieved depends on both the state of knowledge about a system and how well the modelling is done. Examples of the range of objectives are:

1. Developing scientific understanding- through quantitative expression of current knowledge of a system (as well as displaying what we know, this may also show up what we do not know);

2. tests the effect of changes in a system;

3. aids decision making, including: (i) tactical decisions by managers; and (ii) strategic decisions by planners.

1.1.3 Classifications of models

When studying models, it is helpful to identify broad categories of models. Classification of individual models into these categories tells us immediately some of the essentials of their structure.

One division between models is based on the type of outcome they predict. Deterministic models ignore random variation, and so always predict the same outcome from a given starting point. On the other hand, the model may be more statistical in nature and so may predict the distribution of possible outcomes. Such models are said to be stochastic.

A second method of distinguishing between types of models is to consider the level of understanding on which the model is based. The simplest explanation is to consider the hierarchy of organizational structures within the system being modeled.

1.1.4 Model building

Model building involves imagination and skill. Giving rules for doing it is like listing rules for being an artist; at best this provides a framework around which to build skills and develop imagination. The outline of the modeling process is as follows;

Formulate the Problem What is it that you wish to know? The nature of the model you choose depends very much on what you want it to do.

Outline the Model At this stage you must separate the various parts of the universe into unimportant, exogenous, and endogenous. The interrelations among the variables must also be specified.

Is It Useful? Now stand back and look at what you have. Can you obtain the needed data and then use it in the model to make the predictions you want? If the answer is no, then you must reformulate the model (step 2) and perhaps even the problem (step 1). Note that "useful" does not mean reasonable or accurate; they come in step 4. It means: the model fits the situation; will we be able to use it?

Test the Model Use the model to make predictions that can be checked against data or common sense. It is not advisable to rely entirely on common sense, because it may well be wrong. Start out with easy predictions-don't waste time on involved calculations with a model that may be no good. If these predictions are bad and there are no mathematical errors, return to step 2 or step 1. If these predictions are acceptable, they should give you some feeling for the accuracy and range of applicability of the model. If they are less accurate than you anticipated, it is a good idea to try to understand why, since this may uncover implicit or false assumptions.

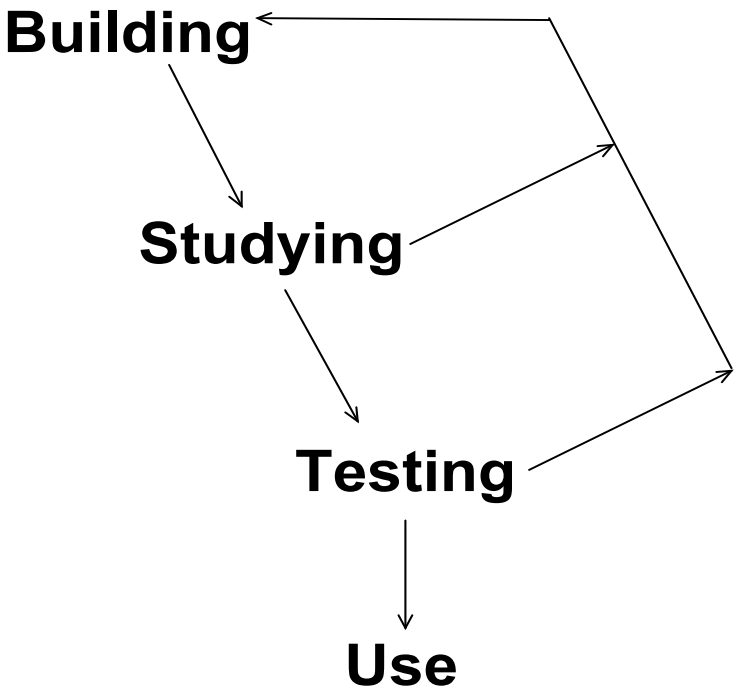


Figure 1.1: *The key components studied under the umbrella of applied mathematics: (1) mathematical science, (2) domain-specific knowledge, and (3) a few 'math-adjacent' disciplines*

1.2 Applied mathematics

1.2.1 What is applied mathematics

We study and develop mathematics as it applies to model, optimize, and control various physical, biological, engineering, and social systems. Applied mathematics is a combination of (1) mathematical science, (2) knowledge and understanding from a particular domain of interest, and often (3) insight from a few 'math-adjacent' disciplines.

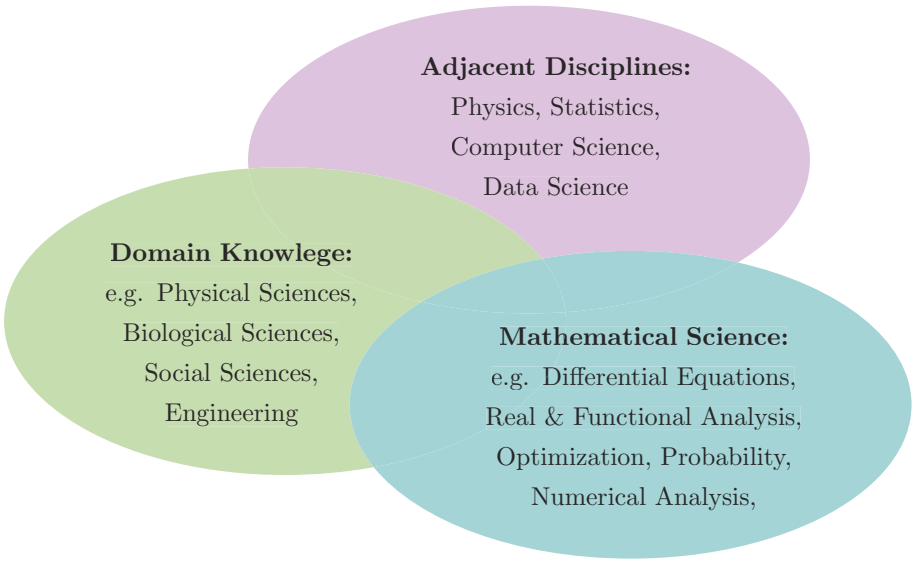


Figure 1.2: *The key components studied under the umbrella of applied mathematics: (1) mathematical science, (2) domain-specific knowledge, and (3) a few 'math-adjacent' disciplines*

Applying mathematics to real-world problems requires mathematical approaches that have evolved to stand up to the many demands and complications of real-world problems. In some applications, a relatively simple set of governing mathematical expressions are able to describe the relevant phenomena. In these situations, problems often require very accurate solutions, and the mathematical challenge is to develop methods that are efficient (and sometimes also adaptable to variable data) without losing accuracy. In other applications, there is no set of governing mathematical expressions (either because we do not know them, or because they may not exist). Here, the challenge is to develop better mathematical descriptions of the phenomena by processes, interpreting and synthesizing imperfect observations.

1. Formulating the problem, first casually, i.e., in terms of standards in sciences and engineering, and then transitioning to a proper mathematical formulation(mathematical modelling);

2. Analyzing the problem by "all means available", including theory, methods, and algorithm toolboxes developed within applied mathematics;

3. Identifying what kinds of solutions are needed, and implementing an appropriate method to find such a solution.

Making contributions to a specific domain that are truly valuable requires more than just mathematical expertise. Domain-specific understanding may change our perspective on what constitutes a solution. For example, whenever system parameters are no longer 'nice' but must be estimated from measurement or experimental data, it becomes more difficult to find meaning in the solutions, and it becomes more important, and challenging, to estimate the uncertainty in solutions. Similarly, whenever a system couples many subsystems at scale, it may be no longer possible to interpret the exact expressions (if they can be computed at all), and approximate, or 'effective' solutions may be more meaningful. In every domain-specific application, it is important to know what problems are most urgent, and what kinds of solutions are most valuable.

Mathematics is not the only field capable of making valuable contributions to other domains, and we think specifically of physics, statistics and computer science as other fields that have each developed their own frameworks, philosophies, and intuitions for describing problems and their solutions. This is particularly evident with the recent developments in data science. The recent deluge of data has brought a wealth of opportunity in engineering, and in the physical, natural and social sciences, where there have been many open problems that could only be addressed empirically. Physics, statistics, and computer science have become fundamental pillars of data science, in part, because each of these 'math-adjacent' disciplines provides a way to analyze and interpret this data constructively. Nonetheless, there are many unresolved challenges ahead, and we believe that a mixture of mathematical insight and some intuition from these adjacent disciplines may help resolve these challenges. Problem Formulation We will rely on a diverse array

of instructional examples from different areas of science and engineering to illustrate how to translate a rather vaguely stated scientific or engineering phenomenon into a crisply stated mathematical challenge. Some of these challenges will be resolved, and some will stay open for further research. We will be referring to instructional examples, such as the Kirchhoff and the Kuramoto-Sivashinsky equations for power systems, the Navier-Stokes equations for fluid dynamics, network flow equations, the Fokker-Planck equation from statistical mechanics, and constrained regression from data science.

Problem Analysis: We analyze problems extracted from applications by all means possible, which requires both domain-specific intuition and mathematical knowledge. We can often make precise statements about the solutions of a problem without actually solving the problem in the mathematical sense. Dimensional analysis from physics is an example of this type of preliminary analysis that is helpful and useful. We may also identify certain properties of the solutions by analyzing any underlying symmetries and establishing the correct principal behaviors expected from the solutions; some important examples involve oscillatory behavior (waves), diffusive behavior, and dissipative/decaying vs. conservative behaviors. One can also extract a lot from analyzing the different asymptotic regimes of a problem, say when a parameter becomes small, making the problem easier to analyze. Matching different asymptotic solutions can give a detailed, even though ultimately incomplete, description.

Solution Construction: As previously mentioned, one component of applied mathematics is a collection of specialized techniques for finding analytic solutions. These techniques are not always feasible, and developing computational intuition should help us to identify proper methods of numerical (or mixed analytic-numerical) analysis, i.e., a specific toolbox, helping to unravel the problem.