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Statistical Inference

Tutorial Sheet 2: Detailed Solutions

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Solution 1 ***Exercise 1: Methods of Estimation***

1. Method of Moments (MoM) Estimator $\tilde{\theta}_n$ *We equate the first theoretical moment to the first empirical moment.*

$$\mathbb{E}[X_1] = \int_0^1 x \cdot (\theta+1)x^\theta \, dx = (\theta+1)\int_0^1 x^{\theta+1}\, dx = (\theta+1)\left[\frac{x^{\theta+2}}{\theta+2}\right]_0^1 = \frac{\theta+1}{\theta+2}$$

Equating to the sample mean $\bar{X}_n$*:*

$$\frac{\tilde{\theta}_n+1}{\tilde{\theta}_n+2} = \bar{X}_n \implies \tilde{\theta}_n+1 = \bar{X}_n\tilde{\theta}_n + 2\bar{X}_n \implies \tilde{\theta}_n(1-\bar{X}_n) = 2\bar{X}_n - 1$$

$$\implies \boxed{\tilde{\theta}_n = \frac{2\bar{X}_n - 1}{1-\bar{X}_n}}$$

2. Maximum Likelihood Estimator (MLE) $\hat{\theta}_n$ *The likelihood function is* $L(\theta) = (\theta+1)^n \left(\prod_{i=1}^n x_i\right)^\theta$*. The log-likelihood is* $\ell(\theta) = n\ln(\theta+1) + \theta\sum_{i=1}^n \ln(x_i)$*. Taking the derivative with respect to* θ *and setting it to zero:*

$$\ell'(\theta) = \frac{n}{\theta+1} + \sum_{i=1}^n \ln(x_i) = 0 \implies \frac{n}{\hat{\theta}_n+1} = -\sum_{i=1}^n \ln(X_i)$$

$$\implies \boxed{\hat{\theta}_n = -\frac{n}{\sum_{i=1}^n \ln(X_i)} - 1}$$

(Verification: $\ell''(\theta) = -\frac{n}{(\theta+1)^2} < 0$*, confirming a global maximum).*

3. Comparison *The two estimators are* ***not identical****. The MoM estimator relies on the arithmetic mean* $\bar{X}_n$*, whereas the MLE relies on the geometric mean (via the sum of logarithms).*

Solution 2 ***Exercise 2: Properties of Estimators***

Preliminary: If $X_i \sim \mathcal{E}(\lambda)$*, the sum* $S_n = \sum_{i=1}^n X_i$ *follows a Gamma distribution* $\Gamma(n,\lambda)$ *with density* $f_{S_n}(s) = \frac{\lambda^n}{(n-1)!}s^{n-1}e^{-\lambda s}\mathbf{1}_{\mathbb{R}^+}(s)$*. The estimators are* $\hat{\lambda}_1 = \frac{n}{S_n}$ *and* $\hat{\lambda}_2 = \frac{n-1}{S_n}$*.*

1. Bias of the estimators *First, calculate* $\mathbb{E}[S_n^{-1}]$ *(valid for* $n > 1$*):*

$$\mathbb{E}[S_n^{-1}] = \int_0^\infty s^{-1}\frac{\lambda^n}{(n-1)!}s^{n-1}e^{-\lambda s}\, ds = \frac{\lambda^n}{(n-1)!}\frac{\Gamma(n-1)}{\lambda^{n-1}} = \frac{\lambda}{n-1}$$

- $\mathbb{E}[\hat{\lambda}_1] = n\mathbb{E}[S_n^{-1}] = \frac{n\lambda}{n-1}$*. Bias:* $Bias(\hat{\lambda}_1) = \frac{n\lambda}{n-1} - \lambda = \boxed{\frac{\lambda}{(n-1)}}$.
- $\mathbb{E}[\hat{\lambda}_2] = (n-1)\mathbb{E}[S_n^{-1}] = (n-1)\frac{\lambda}{n-1} = \lambda$*. Bias:* $\boxed{0}$*.* $\hat{\lambda}_2$ ***is unbiased.***

2. Variance of the unbiased estimator *Calculate* $\mathbb{E}[S_n^{-2}]$ *(valid for* $n > 2$*):*

$$\mathbb{E}[S_n^{-2}] = \int_0^\infty s^{-2} \frac{\lambda^n}{(n-1)!} s^{n-1} e^{-\lambda s}\, ds = \frac{\lambda^n}{(n-1)!} \frac{\Gamma(n-2)}{\lambda^{n-2}} = \frac{\lambda^2}{(n-1)(n-2)}$$

$$Var(\hat{\lambda}_2) = \mathbb{E}[\hat{\lambda}_2^2] - (\mathbb{E}[\hat{\lambda}_2])^2 = (n-1)^2 \frac{\lambda^2}{(n-1)(n-2)} - \lambda^2 = \boxed{\frac{\lambda^2}{n-2}}$$

3. Mean Squared Error (MSE)

- $MSE(\hat{\lambda}_2) = Var(\hat{\lambda}_2) = \frac{\lambda^2}{n-2}$.
- $Var(\hat{\lambda}_1) = n^2 \left(\mathbb{E}[S_n^{-2}] - (\mathbb{E}[S_n^{-1}])^2\right) = n^2 \left(\frac{\lambda^2}{(n-1)(n-2)} - \frac{\lambda^2}{(n-1)^2}\right) = \frac{n^2\lambda^2}{(n-1)^2(n-2)}$.
- $MSE(\hat{\lambda}_1) = \ Var(\hat{\lambda}_1) + Bias^2 = \frac{n^2\lambda^2}{(n-1)^2(n-2)} + \frac{\lambda^2}{(n-1)^2} = \frac{\lambda^2(n+2)}{(n-1)(n-2)}$.

Conclusion: *Since* $\frac{n+2}{(n-1)(n-2)} > \frac{n-1}{(n-1)(n-2)} = \frac{1}{n-2}$*, we have* $MSE(\hat{\lambda}_1) > MSE(\hat{\lambda}_2)$*. Thus,* $\hat{\lambda}_2$ ***is the better estimator*** *in terms of MSE.*

Solution 3 ***Exercise 3: Fisher Information and Cramér-Rao Bound***

1. Score Function and Fisher Information *Log-probability mass function:* $\ln f(x;\lambda) = -\lambda + x\ln\lambda - \ln(x!)$*. Score function:* $S(\lambda;x) = \frac{\partial \ln f}{\partial \lambda} = -1 + \frac{x}{\lambda} = \frac{x-\lambda}{\lambda}$*. Fisher information:* $I(\lambda) = Var(S(\lambda;X)) = Var\left(\frac{X-\lambda}{\lambda}\right) = \frac{1}{\lambda^2} Var(X) = \frac{\lambda}{\lambda^2} = \boxed{\frac{1}{\lambda}}$.

2. Sample Fisher Information *Since observations are i.i.d.,* $I_n(\lambda) = n \cdot I(\lambda) = \boxed{\frac{n}{\lambda}}$.

3. Cramér-Rao Lower Bound (CRLB) *For any unbiased estimator* T *of* λ*:* $Var(T) \geq \frac{1}{I_n(\lambda)} = \boxed{\frac{\lambda}{n}}$.

4. Efficiency of $\bar{X}_n$ $\mathbb{E}[\bar{X}_n] = \lambda$ *(unbiased) and* $Var(\bar{X}_n) = \frac{\lambda}{n}$*. Since its variance exactly achieves the CRLB,* $\bar{X}_n$ ***is an efficient estimator*** *of* λ*.*

Solution 4 ***Exercise 4: Sufficiency***

1. Joint density $f(x_1,\ldots,x_n;\theta) = \prod_{i=1}^n \frac{1}{\theta}\mathbf{1}_{[0,\theta]}(x_i) = \frac{1}{\theta^n}\left(\prod_{i=1}^n \mathbf{1}_{x_i\geq 0}\right)\mathbf{1}_{\max(x_i)\leq\theta}$.

2. Sufficiency of $T = \max(X_1,\ldots,X_n)$ *By the Fisher-Neyman Factorization Theorem, let* $g(t;\theta) = \frac{1}{to\theta^n}\mathbf{1}_{t\leq\theta}$ *and* $h(x_1,\ldots,x_n) = \prod_{i=1}^n \mathbf{1}_{x_i\geq 0}$*. Since* $f(x;\theta) = g(T(x);\theta)h(x)$*,* T ***is a sufficient statistic for*** θ*.*

3. Sufficiency of $T' = \bar{X}_n$ ***No.*** *The support of the joint density depends on* θ *via* $\max(x_i) \leq \theta$*. The sample mean* $\bar{X}_n$ *cannot capture this support constraint, making factorization impossible using only* $\sum x_i$*.*

4. Sufficiency of $T'' = X_1$ ***No.*** X_1 *discards the information contained in* $X_2,\ldots,X_n$*. The conditional distribution of* $(X_2,\ldots,X_n)$ *given* X_1 *is the product of* $n-1$ *independent* $\mathcal{U}(0,\theta)$ *distributions, which explicitly depends on* θ*.*

Solution 5 ***Exercise 5: Completeness and Lehmann-Scheffé Theorem***

1. Unbiased estimator U ***of*** $p(1-p)$ *Since* $X_i \in \{0,1\}$*,* $X_1(1-X_2) = 1$ *iff* $X_1 = 1$ *and* $X_2 = 0$*.* $\mathbb{E}[U] = P(X_1 = 1, X_2 = 0) = P(X_1 = 1)P(X_2 = 0) = p(1-p)$*. Thus,* U ***is an unbiased estimator of*** $p(1-p)$*.*

2. Calculation of $U^* = \mathbb{E}[U \mid T]$ *Given* $T = \sum_{i=1}^n X_i = t$*, the vector* $(X_1,\ldots,X_n)$ *is uniformly distributed over all* $\binom{n}{t}$ *binary configurations summing to* t*.*

- $\mathbb{E}[X_1 \mid T = t] = P(X_1 = 1 \mid T = t) = \frac{\binom{n-1}{t-1}}{\binom{n}{t}} = \frac{t}{n}$.

- $\mathbb{E}[X_1X_2 \mid T=t] = P(X_1=1, X_2=1 \mid T=t) = \frac{\binom{n-2}{t-2}}{\binom{n}{t}} = \frac{t(t-1)}{n(n-1)}$.

Therefore, $U^*(t) = \frac{t}{n} - \frac{t(t-1)}{n(n-1)} = \frac{t(n-1)-t(t-1)}{n(n-1)} = \boxed{\frac{t(n-t)}{n(n-1)}}$.

3. Conclusion *T is complete and sufficient. U^* is a function of T and is unbiased for $p(1-p)$. By the* ***Lehmann-Scheffé theorem****, $U^* = \frac{T(n-T)}{n(n-1)}$ is the* ***Uniformly Minimum Variance Unbiased Estimator (UMVUE)*** *of $p(1-p)$.*

Solution 6 ***Exercise 6: Synthesis – Exponential Family***

1. Exponential Family *$f(x;\alpha) = \exp\left(\ln(\alpha) - (\alpha+1)\ln(x)\right)\mathbf{1}_{x>1} = \exp\left(-\alpha\ln(x) + \ln(\alpha)\right)\mathbf{1}_{x>1}$. This matches the canonical form $\exp(\eta(\alpha)T(x) - A(\alpha) + B(x))\mathbf{1}_{x\in S}$ with $\eta(\alpha) = -\alpha$ and $T(x) = \ln(x)$.* ***It belongs to the exponential family.***

2. Complete and Sufficient Statistic *The parameter space $\alpha \in]0, +\infty[$ contains an open set, so the family is of full rank. The natural sufficient statistic is $T_n = \sum_{i=1}^n \ln(X_i)$. T_n* ***is complete and sufficient for α.***

3. Maximum Likelihood Estimator *$\ell(\alpha) = n\ln(\alpha) - (\alpha+1)\sum_{i=1}^n \ln(x_i)$. $\ell'(\alpha) = \frac{n}{\alpha} - \sum_{i=1}^n \ln(x_i) = 0 \implies \boxed{\hat{\alpha}_n = \frac{n}{\sum_{i=1}^n \ln(X_i)}}$.*

4. Bias and Unbiased Estimator *Let $Y_i = \ln(X_i)$. For $y > 0$, $P(Y_i \le y) = P(X_i \le e^y) = \int_1^{e^y} \alpha x^{-\alpha-1}dx = 1 - e^{-\alpha y}$. Thus, $Y_i \sim \mathcal{E}(\alpha)$, and $S_n = \sum_{i=1}^n Y_i \sim \Gamma(n, \alpha)$. The MLE is $\hat{\alpha}_n = \frac{n}{S_n}$. From Exercise 2, $\mathbb{E}[S_n^{-1}] = \frac{\alpha}{n-1}$ (for $n > 1$). $\mathbb{E}[\hat{\alpha}_n] = n\mathbb{E}[S_n^{-1}] = \frac{n\alpha}{n-1}$. Bias: $\text{Bias}(\hat{\alpha}_n) = \frac{n\alpha}{n-1} - \alpha = \boxed{\frac{\alpha}{n-1}}$.* ***The MLE is biased.*** *To make it unbiased, set $\tilde{\alpha}_n = c\hat{\alpha}_n$. We need $c\frac{n\alpha}{n-1} = \alpha \implies c = \frac{n-1}{n}$.* ***The unbiased estimator is:*** $\boxed{\tilde{\alpha}_n = \frac{n-1}{\sum_{i=1}^n \ln(X_i)}}$. *(Note: By Lehmann-Scheffé, this is also the UMVUE of α).*