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Statistical Inference

Tutorial Sheet 2: Point Estimation

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Exercise 1 *Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sample from a distribution with the probability density function:*

$$f(x;\theta) = (\theta+1)x^{\theta}\mathbf{1}_{]0,1[}(x)$$

where $\theta > -1$ is an unknown parameter.

1. *Find the Method of Moments (MoM) estimator $\tilde{\theta}_n$ for the parameter θ.*
2. *Find the Maximum Likelihood Estimator (MLE) $\hat{\theta}_n$.*
3. *Are the two estimators identical? Briefly justify your answer.*

Exercise 2 *Let $X_1, \ldots, X_n$ be an i.i.d. sample from an exponential distribution $\mathcal{E}(\lambda)$ with density*

$$f(x;\lambda) = \lambda e^{-\lambda x}\mathbf{1}_{\mathbb{R}^+}(x).$$

Consider the following two estimators for the parameter λ:

$$\hat{\lambda}_1 = \frac{1}{\bar{X}_n} \quad \text{and} \quad \hat{\lambda}_2 = \frac{n-1}{\sum_{i=1}^{n} X_i}$$

1. *Calculate the bias of $\hat{\lambda}_1$ and $\hat{\lambda}_2$. Which one is unbiased?*
2. *Calculate the variance of the unbiased estimator (assume $n > 2$).*
3. *Calculate the Mean Squared Error (MSE) of both estimators (assume $n > 2$). Which one is "better" in terms of MSE?*

Exercise 3 *Let $X_1, \ldots, X_n$ be an i.i.d. sample from a Poisson distribution $\mathcal{P}(\lambda)$, with $\lambda > 0$.*

1. *Calculate the score function $S(\lambda; X_1)$ and the Fisher information $I(\lambda)$ for a single observation.*
2. *Deduce the Fisher information $I_n(\lambda)$ for the entire sample.*
3. *State the Cramér-Rao Lower Bound (CRLB) for the variance of any unbiased estimator of λ.*
4. *Show that the sample mean $\bar{X}_n$ is an efficient estimator of λ.*

Exercise 4 *Let $X_1, \ldots, X_n$ be an i.i.d. sample from a uniform distribution $\mathcal{U}(0,\theta)$, where $\theta > 0$ is unknown.*

1. *Write down the joint probability density function of the sample.*
2. *Using the Fisher-Neyman Factorization Theorem, show that $T = \max(X_1, \ldots, X_n)$ is a sufficient statistic for θ.*

3. *Is the statistic $T' = \bar{X}_n$ sufficient for θ? Justify briefly.*

4. *Is the statistic $T'' = X_1$ sufficient? Justify briefly.*

Exercise 5 *Let $X_1, \ldots, X_n$ be an i.i.d. sample from a Bernoulli distribution $\mathcal{B}(1, p)$, with $p \in]0, 1[$. Recall that $T = \sum_{i=1}^{n} X_i$ is a complete and sufficient statistic for p.*

1. *We wish to estimate the variance of the distribution, $g(p) = p(1-p)$. Show that $U = X_1(1 - X_2)$ is an unbiased estimator of $p(1-p)$.*

2. *Using the Rao-Blackwell and Lehmann-Scheffé theorems, compute the improved estimator $U^* = \mathbb{E}[U \mid T]$.*

3. *Conclude on the statistical nature of U^*.*

Exercise 6 *Let $X_1, \ldots, X_n$ be an i.i.d. sample from a Pareto distribution with density:*

$$f(x; \alpha) = \frac{\alpha}{x^{\alpha+1}} \mathbf{1}_{]1, +\infty[}(x)$$

where $\alpha > 0$ is an unknown parameter.

1. *Show that this distribution belongs to the exponential family.*

2. *Deduce a complete and sufficient statistic T_n for α.*

3. *Determine the Maximum Likelihood Estimator (MLE) $\hat{\alpha}_n$.*

4. *Is $\hat{\alpha}_n$ unbiased? If not, propose an unbiased estimator $\tilde{\alpha}_n$ based on $\hat{\alpha}_n$.*