Chapter III

Strain Tensor

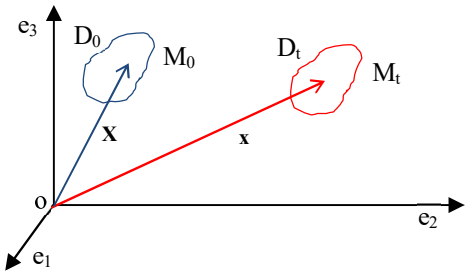
III-1 Motion and its representations

Let D_0 be a material domain contained within a material system and equipped with a fixed reference frame R(O, e_1, e_2, e_3).

D_0 : reference configuration at time t_0

M_0 :position of a point M at time t_0

$$\overrightarrow{OM_0} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \quad \text{or} \quad \overrightarrow{OM_0} = \vec{X} = X_P\, e_P$$



X_p : Material coordinates (referential coordinates)

D_t : current configuration at time t

M_t : position of a point M at time t

$$\overrightarrow{OM_t} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad \overrightarrow{OM_t} = \vec{x} = x_q\, e_q$$

x_q : Spatial coordinates

Let χ be the function such that : $(M_0, t) \xrightarrow{\chi} \overrightarrow{OM_t}$ with $\begin{cases} \overrightarrow{OM_t} = \chi(M_0, t) = \chi(\vec{X}, t) \\ or \quad \vec{x} = \chi(\vec{X}, t) = \vec{x}(\vec{X}, t) \end{cases}$

In the Lagrangian representation (or Material description) of motion:

Variables : X_1 , X_2 , X_3 ,t or X_i ,t

Unknowns: $x_i = x_i(X_i, t) = \chi_i(X_i, t)$, i=1, 2, 3

The velocity of a material point M at time t is defined by :

$$\vec{V}(\vec{X}, t) = \left.\frac{\partial \chi}{\partial t}\right|_{\vec{X}} (\vec{X}, t) = \left.\frac{\partial \vec{x}}{\partial t}\right|_{\vec{X}} (\vec{X}, t)$$

The acceleration of a material point M at time t is defined by :

$$\vec{\Gamma}(\vec{X},t)=\frac{\partial \mathrm{V}}{\partial t}\Big|_{\vec{X}}(\vec{X},t)=\frac{\partial^2\chi}{\partial t^2}\Big|_{\vec{X}}(\vec{X},t)=\frac{\partial^2\vec{x}}{\partial t^2}\Big|_{\vec{X}}(\vec{X},t)$$

In the Eulerian representation (or Spatial description) of motion:

Variables : x_1 , x_2 , x_3 ,t or x_i ,t

Unknowns: $v_i = v_i(x_i, t) = v_i(x_1, x_2, x_3, t)$, i=1, 2, 3

Note: determining the (Lagrangian) velocity $\vec{V}(\vec{X},t)$ requires knowledge of the initial position $\vec{X}$ and the time t, whereas determining the (Eulerian) velocity $\vec{v}(\vec{x},t)$ requires knowledge of the current position $\vec{x}$ and the time t.

At any instant, the two velocities (Lagrangian and Eulerian) must be equal :

$$\vec{V}(\vec{X},t)=\frac{\partial\chi}{\partial t}\Big|_{\vec{X}}(\vec{X},t)=\vec{v}(\vec{x}(\vec{X},t),t)$$

III-2 Material Derivative

The material (or substantial) derivative of a quantity A associated with a material particle represents the variation of that quantity over time as one follows the particle in its motion.

In the Lagrangian description: $A=A(\vec{X},t)$

The material derivative is simply the partial derivative with respect to time t:

$$\frac{d\mathrm{A}}{dt}\Big|_{\vec{X}}(\vec{X},t)=\frac{\partial A(\vec{X},t)}{\partial t}\Big|_{\vec{X}}(\vec{X},t)$$

In the Eulerian description: $A=A(\vec{x},t)$

$$\frac{d\mathrm{A}}{dt}(\vec{x},t)=\frac{\partial A(\vec{x},t)}{\partial t}\Big|_{\vec{x}}+\frac{\partial A(\vec{x},t)}{\partial x_i}\cdot\frac{\partial x_i}{\partial t}$$

$$\frac{d\mathrm{A}}{dt}(\vec{x},t)=\frac{\partial A(\vec{x},t)}{\partial t}\Big|_{\vec{x}}+\frac{\partial A(\vec{x},t)}{\partial x_1}\cdot\frac{\partial x_1}{\partial t}+\frac{\partial A(\vec{x},t)}{\partial x_2}\cdot\frac{\partial x_2}{\partial t}+\frac{\partial A(\vec{x},t)}{\partial x_3}\cdot\frac{\partial x_3}{\partial t}$$

$$\frac{d\mathrm{A}}{dt}(\vec{x},t)=\frac{\partial A(\vec{x},t)}{\partial t}\Big|_{\vec{x}}+\frac{\partial A(\vec{x},t)}{\partial x_1}\cdot v_1+\frac{\partial A(\vec{x},t)}{\partial x_2}\cdot v_2+\frac{\partial A(\vec{x},t)}{\partial x_3}\cdot v_3$$

$$\frac{d\mathrm{A}}{dt}(\vec{x},t)=\frac{\partial A(\vec{x},t)}{\partial t}\Big|_{\vec{x}}+grad(A(\vec{x},t))\cdot\vec{v}(\vec{x},t)$$

Note: The quantity A can be scalar, vectorial, or tensorial.

Example: acceleration

In the Lagrangian description: $\vec{V} = \vec{V}(\vec{X}, t)$

$$\vec{\Gamma}(\vec{X}, t) = \frac{d\vec{V}}{dt}\Big|_{\vec{X}}(\vec{X}, t) = \frac{\partial \vec{V}(\vec{X}, t)}{\partial t}\Big|_{\vec{X}}(\vec{X}, t)$$

In the Eulerian description: $\vec{v} = \vec{v}(\vec{x}, t)$

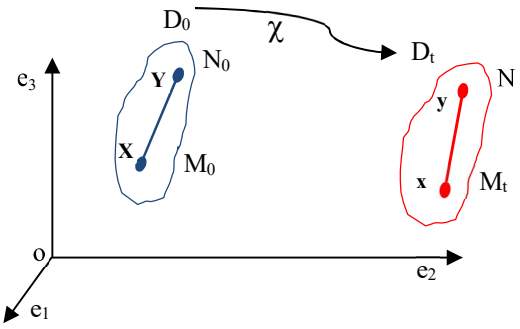
$$\vec{\gamma}(\vec{x}, t) = \frac{d\vec{v}}{dt}(\vec{x}, t) = \frac{\partial \vec{v}(\vec{x}, t)}{\partial t}\Big|_{\vec{x}}(\vec{x}, t) + \frac{\partial \vec{v}(\vec{x}, t)}{\partial x_i} \cdot \frac{\partial x_i}{\partial t}$$

$$\vec{\gamma}(\vec{x}, t) = \frac{d\vec{v}}{dt}(\vec{x}, t) = \frac{\partial \vec{v}(\vec{x}, t)}{\partial t}\Big|_{\vec{x}}(\vec{x}, t) + grad(\vec{v}(\vec{x}, t)) \cdot \vec{v}(\vec{x}, t)$$

or $\quad \gamma_j(\vec{x}, t) = \frac{dv_j}{dt}(\vec{x}, t) = \frac{\partial v_j(\vec{x}, t)}{\partial t}\Big|_{\vec{x}}(\vec{x}, t) + \frac{\partial v_j(\vec{x}, t)}{\partial x_i} \cdot \frac{\partial x_i}{\partial t} = v_{j,t} + v_{j,i} \cdot v_i$

III-3 Transformation gradient tensor

Consider two points M_0 and N_0 such that:



$\overrightarrow{OM_0} = (X_1, X_2, X_3)$ or $\overrightarrow{OM_0} = \vec{X} = X_P\, e_P$

$\overrightarrow{OM_t} = (x_1, x_2, x_3)$ or $\overrightarrow{OM_t} = \vec{x} = x_q\, e_q$

with $\overrightarrow{OM_t} = \vec{x} = \chi(\vec{X}, t) = \vec{x}(\vec{X}, t)$

and

$\overrightarrow{ON_0} = (Y_1, Y_2, Y_3)$ or $\overrightarrow{ON_0} = \vec{Y} = Y_p\, e_P$

$\overrightarrow{ON_t} = (y_1, y_2, y_3)$ or $\overrightarrow{ON_t} = \vec{y} = y_q\, e_q$

with $\overrightarrow{ON_t} = \vec{y} = \chi(\vec{Y}, t) = \vec{y}(\vec{Y}, t)$

thus we have $\overrightarrow{M_0N_0} = \vec{Y} - \vec{X}$ and $\overrightarrow{M_tN_t} = \vec{y} - \vec{x}$

By applying the limited development (Taylor expansion) in the neighborhood of $M_0(X_1, X_2, X_3)$:

$$y_i - x_i = \frac{\partial \chi_i}{\partial X_P}(X_1, X_2, X_3, t)(Y_p - X_p) + [(Y_1 - X_1)^2 + (Y_2 - X_2)^2 + (Y_3 - X_3)^2]^{\frac{1}{2}} \propto_i (Y_1 - X_1, Y_2 - X_2, Y_3 - X_3, t)$$

$\propto_i \to 0$ if $Y_i \to X_i \quad (N_0 \to M_0)$

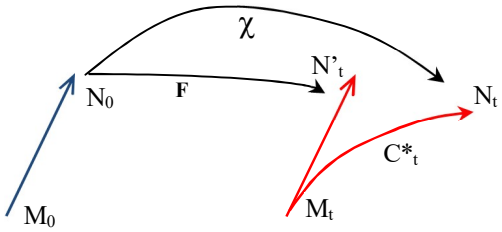
The vector form: $\vec{y} - \vec{x} = F(\vec{X}, t)(\vec{Y} - \vec{X}) + \|\vec{Y} - \vec{X}\| \, \vec{\alpha}(\vec{Y} - \vec{X}, t)$

$F_{ij} = \frac{\partial \chi_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j}$ is a second-order tensor called the gradient tensor of the transformation

$$\mathrm{F} = \mathrm{grad}(\chi) = \mathrm{grad}\,(\mathrm{x}) = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{pmatrix}$$

$\overrightarrow{M_t N_t} = F(M_0, t)\,\overrightarrow{M_0 N_0} + \|\overrightarrow{M_0 N_0}\| \, \vec{\alpha}\,(\overrightarrow{M_0 N_0}, t)$

$\overrightarrow{M_t N_t} = \overrightarrow{M_t N_t'} + \|\overrightarrow{M_0 N_0}\| \, \vec{\alpha}\,(\overrightarrow{M_0 N_0}, t)$



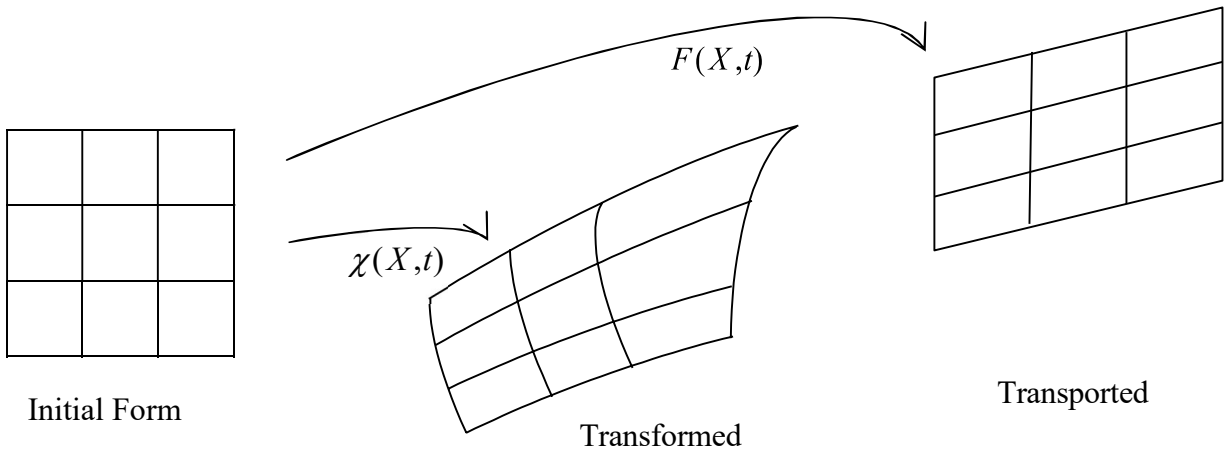
C_t^* : transformed vector of $\overrightarrow{M_0 N_0}$

$\overrightarrow{M_t N_t'} = F(M_0, t)\,\overrightarrow{M_0 N_0}$: transported vector of $\overrightarrow{M_0 N_0}$

Let $\overrightarrow{M_0 N_0} = \overrightarrow{dX}$ and $\overrightarrow{M_t N_t'} = \overrightarrow{dx}$ ===> $\overrightarrow{dx} = F \cdot \overrightarrow{dX}$ or $dx_i = \frac{\partial x_i}{\partial X_P} dX_p$

Note: The Jacobian J is defined as the determinant of the Jacobian matrix:

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{vmatrix} = det(F)$$ notation $J(X,t) = \frac{D(x_1, x_2, x_3)}{D(X_1, X_2, X_3)} = det(F)$



III-4 Stretch tensor

Let $\overrightarrow{dx} = F \cdot \overrightarrow{dX}$ and $\overrightarrow{dx'} = F \cdot \overrightarrow{dX'}$

$$\overrightarrow{dx} \cdot \overrightarrow{dx'} = F \cdot \overrightarrow{dX} \cdot F \cdot \overrightarrow{dX'} = \overrightarrow{dX} \cdot F^T F \cdot \overrightarrow{dX'}$$

$$= \overrightarrow{dX} \cdot C \cdot \overrightarrow{dX'}$$

$C = F^T F$: is called Cauchy-Green Stretch Tensor at M_0 point and at time t.

It's a Lagrangian tensor. We also have:

$$C_{ij} = F_{pi} \cdot F_{pj} = \left(\frac{\partial x_p}{\partial X_i}\right) \cdot \left(\frac{\partial x_p}{\partial X_j}\right)$$

The Stretch $\lambda(\vec{u}_0)$ in a direction $\vec{u}_0$ is defined by the relation:

$$\lambda(\vec{u}_0) = \frac{d\ell}{d\ell_0} = [\vec{u}_0 \, C \, \vec{u}_0]^{1/2} = [C(\vec{u}_0, \vec{u}_0]^{1/2}$$

($\vec{u}_0$: unit vector)

III-5 Strain tensor

$$\overrightarrow{dx} \cdot \overrightarrow{dx'} - \overrightarrow{dX} \cdot \overrightarrow{dX'} = \overrightarrow{dX} \cdot C \cdot \overrightarrow{dX'} - \overrightarrow{dX} \cdot \overrightarrow{dX'} = \overrightarrow{dX} \cdot (C - I) \cdot \overrightarrow{dX'} = 2\overrightarrow{dX} \cdot \frac{(C-I)}{2} \cdot \overrightarrow{dX'}$$

$$= 2 \cdot \overrightarrow{dX} \cdot E \cdot \overrightarrow{dX'}$$

$E = \frac{1}{2}(C - I)$: is called Cauchy-Green Stretch Tensor at M_0 point and at time t.

It's a Lagrangian tensor. We also have:

$$E_{ij} = \frac{1}{2}(C_{ij} - \delta_{ij})$$ (δ_{ij}: Kronecker symbol)

The Strain $E(\vec{u}_0, \vec{u}_0)$ in a direction $\vec{u}_0$ is defined by the relation:

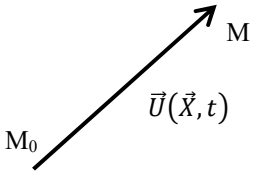
$$E(\vec{u}_0, \vec{u}_0) = \frac{1}{2}\frac{(d\ell)^2 - (d\ell_0)^2}{(d\ell_0)^2} = [\vec{u}_0 \, E \, \vec{u}_0]$$

Thus $$E(\vec{u}_0, \vec{u}_0) = \frac{1}{2}[\lambda(\vec{u}_0)^2 - 1]$$ and $$\lambda(\vec{u}_0) = [1 + 2 \cdot E(\vec{u}_0, \vec{u}_0)]^{\frac{1}{2}}$$

The Relative Elongation $\delta(\vec{u}_0)$ in a direction $\vec{u}_0$ is defined by the relation:

$$\delta(\vec{u}_0) = \frac{d\ell - d\ell_0}{d\ell_0} = \lambda(\vec{u}_0) - 1 = [1 + 2 \cdot E(\vec{u}_0, \vec{u}_0)]^{\frac{1}{2}} - 1$$

III-6 Displacement gradient tensor



$$\vec{U}(\vec{X},t) = \overrightarrow{M_0M} = \overrightarrow{OM} - \overrightarrow{OM_0} = \vec{x}(\vec{X},t) - \vec{X}$$

$$U_i = x_i - X_i \qquad \Rightarrow \qquad H_{ij} = \frac{\partial U_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j} - \delta_{ij}$$

H = Grad(U) = F – I : is called Displacement gradient tensor

With $\vec{U}(\vec{X},t) = \vec{x}(\vec{X},t) - \vec{X}$: Displacement vector

The displacement gradient tensor H can be uniquely decomposed into a symmetric part and an antisymmetric part.

$H = \mathscr{E} + \Omega = H^s + H^a$

with $\mathscr{E} = sym(H) = \frac{1}{2}(H + H^T) \qquad \Rightarrow \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right)$

$\Omega = antisym(H) = \frac{1}{2}(H - H^T) \qquad \Rightarrow \quad \omega_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} - \frac{\partial U_j}{\partial X_i}\right)$

Note : $C = F^T F = (H+I)^T (H+I) = I + 2\mathscr{E} + H^T \cdot H$

$$E = \frac{1}{2}(C - I) = \mathscr{E} + \frac{1}{2}H^T \cdot H$$

III-7 Small-deformation theory

The strain tensor is a non-linear function of the displacement field.

$$E = \frac{1}{2}(C - I) = \mathscr{E} + \frac{1}{2}H^T \cdot H = \frac{1}{2}(H + H^T) + \frac{1}{2}H^T \cdot H$$

This tensor can be linearized under the small-perturbation hypothesis (SPH). This hypothesis stipulates that displacements are small relative to the reference dimensions of the domain and that the terms of the displacement gradient are small compared to unity. In this case, the current configuration is assumed to coincide with the reference configuration (F ≈ I).

$$E \cong \mathscr{E} = sym(H) = \frac{1}{2}(H + H^T) \qquad E_{ij} \cong \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right) = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

$\mathscr{E}$: Linearized strain tensor

Thus, the stretch $\lambda(\vec{u}_0)$ and the unit elongation $\delta(\vec{u}_0)$ in a direction $\vec{u}_0$ will be:

$$\lambda(\vec{u}_0) \cong 1 + \mathscr{E}(\vec{u}_0, \vec{u}_0) = 1 + \vec{u}_0 \mathscr{E} \vec{u}_0$$

$$\delta(\vec{u}_0) \cong \mathscr{E}(\vec{u}_0, \vec{u}_0) = \vec{u}_0 \mathscr{E} \vec{u}_0$$

III-8 Principal strains (elongations) and principal directions

We consider the linearized strain tensor $\mathscr{E}$

$$\mathscr{E} = sym(H) = \frac{1}{2}(H + H^T) = \frac{1}{2}\left[\mathrm{Grad}(U) + \mathrm{Grad}(U)^T\right]$$

$$\mathscr{E} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix} \qquad \text{with} \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

This second-order strain tensor is symmetric; therefore, there exists an orthonormal eigenbasis (known as the principal basis) in which the tensor is diagonal. In this basis, denoted by (X_1, X_2, X_3), this tensor takes the following form:

$$\mathscr{E} = \begin{pmatrix} \varepsilon_I & 0 & 0 \\ 0 & \varepsilon_{II} & 0 \\ 0 & 0 & \varepsilon_{III} \end{pmatrix}$$

The principal strains (or elongations) ε_I, ε_{II} and ε_{III} represent the eigenvalues of the matrix ($\mathscr{E}$) which can be obtained by solving the following equation:

$$\det(\mathscr{E} - \lambda \mathrm{I}) = 0$$

Expanding the determinant produces a cubic equation in terms of λ *(called characteristic equation)*:

$$det[\mathscr{E} - \lambda I] = -\lambda^3 + I_a\lambda^2 - I_B\lambda + I_C = 0$$

The roots of the characteristic equation determine the allowable values for λ ($\lambda_1, \lambda_2, \lambda_3$) such as :

$\varepsilon_I = \lambda_1$ $\varepsilon_{II} = \lambda_2$ $\varepsilon_{III} = \lambda_3$

Where the scalars I_a, I_b, and I_c are called the *fundamental invariants of the tensor* $\mathscr{E}$ such as :

$$I_a = tr(\mathscr{E}) = \varepsilon_{ii} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \lambda_1 + \lambda_2 + \lambda_3$$

$$I_b = \tfrac{1}{2}(\varepsilon_{ii}\varepsilon_{jj} - \varepsilon_{ij}\varepsilon_{ij}) = \begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{vmatrix} + \begin{vmatrix} \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{23} & \varepsilon_{33} \end{vmatrix} + \begin{vmatrix} \varepsilon_{11} & \varepsilon_{13} \\ \varepsilon_{13} & \varepsilon_{33} \end{vmatrix} =$$

$$= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$$

$$I_c = \det(\mathscr{E}) = det\left[\varepsilon_{ij}\right] = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$$

Each of the values of λ (or ε_i) can be substituted into the following vector relation to find the principal directions X_1 , X_2 et X_3 :

$$(\mathscr{E} - \varepsilon_i I)\,\vec{X}_i = \vec{0}$$

The unit vectors X_1, X_2 and X_3 of the principal basis verify the following vector relations:

$$\begin{cases} X_1 = X_2 \wedge X_3 \\ X_2 = X_3 \wedge X_1 \\ X_3 = X_1 \wedge X_2 \end{cases}$$

III-9 Compatibility equations

The components of the strain tensor can be determined by differentiating the displacement field. Deriving the displacement field from the strain tensor is not as straightforward. The components of the strain tensor must satisfy six conditions—known as compatibility or integrability conditions—which are expressed in the following form:

$$2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = \frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2}$$

$$2\frac{\partial^2 \varepsilon_{13}}{\partial x_1 \partial x_3} = \frac{\partial^2 \varepsilon_{11}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_1^2}$$

$$2\frac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} = \frac{\partial^2 \varepsilon_{22}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_2^2}$$

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_1}\left(\frac{\partial \varepsilon_{12}}{\partial x_3} + \frac{\partial \varepsilon_{13}}{\partial x_2} - \frac{\partial \varepsilon_{23}}{\partial x_1}\right)$$

$$\frac{\partial^2 \varepsilon_{22}}{\partial x_1 \partial x_3} = \frac{\partial}{\partial x_2}\left(\frac{\partial \varepsilon_{12}}{\partial x_3} + \frac{\partial \varepsilon_{23}}{\partial x_1} - \frac{\partial \varepsilon_{13}}{\partial x_2}\right)$$

$$\frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_3}\left(\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{13}}{\partial x_2} - \frac{\partial \varepsilon_{12}}{\partial x_3}\right)$$

Tensor form of the compatibility equations:

$$\text{grad}\ \overrightarrow{\text{div}}\ (\mathscr{E}) + \text{grad}^t\ \overrightarrow{\text{div}}\ (\mathscr{E}) - \text{grad grad}\ (\text{tr}(\mathscr{E})) - \Delta\mathscr{E} = 0 \quad (6 \text{ equations})$$