Mathematical Analysis

Chapter 1: Functions of One Variable

- **Domain of definition**
- **Limits**
- **Continuity**
- **Derivatives**
- **Integrals**

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Abstract

In this comprehensive chapter, we study the fundamental notions of differential and integral calculus: limits, continuity, differentiability, the Mean Value Theorems, standard rules of differentiation, techniques of integration (including substitution, integration by parts, and partial fraction decomposition of rational functions). In particular, this expanded edition features exhaustive, rigorous treatments of all seven indeterminate limit forms, providing step-by-step solved examples for L'Hôpital's rule. Every definition is accompanied by complete, step-by-step worked examples.

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Chapter 1

Mathematical Analysis

Mathematical Notations and Symbols

Before developing the core concepts, the following table summarizes the key mathematical logic symbols used throughout this chapter:

Symbol	Mathematical Name	Explanation / Meaning
$\forall$	Universal quantifier	"For all" or "For every"
$\exists$	Existential quantifier	"There exists" or "For at least one"
$\in$	Set membership	"Belongs to" or "Element of"
$\Rightarrow$	Logical implication	"Implies" or "If ... then ..."
$\Leftrightarrow$	Logical equivalence	"Equivalent to" or "If and only if"
iff	Abbreviation	"If and only if" (bidirectional implication)
$\mathbb{R}$	Set of real numbers	The continuum of real numbers $(-\infty, +\infty)$
$\mathbb{R}^*$	Non-zero real numbers	$\mathbb{R} \setminus \{0\}$
D_f	Domain of definition	The set of inputs x for which $f(x)$ is well-defined

Table 1.1: Standard symbols used in mathematical analysis.

1.1 Functions of One Variable

1.1.1 General Definition

A **function** is a relation that associates each element of a set called the *starting set* (domain) with at most one element of another set called the *ending set* (codomain):

$$\begin{aligned} f : D &\longrightarrow A \\ x &\longmapsto f(x) \end{aligned}$$

Definition: Domain of Definition (D_f)

Let $f : \mathbb{R} \to \mathbb{R}$. The **domain of definition** D_f is the subset of real numbers for which the algebraic expression $f(x)$ represents a valid real quantity:

$$D_f = \{x \in \mathbb{R} \mid f(x) \in \mathbb{R}\}.$$

Expression Type	**Algebraic Form**	**Condition for Real Existence**
Rational fraction	$\frac{P(x)}{Q(x)}$	$Q(x) \neq 0$
Even root (Radical)	$\sqrt{A(x)}$	$A(x) \geq 0$
Strict radical denominator	$\frac{1}{\sqrt{A(x)}}$	$A(x) > 0$
Natural logarithm	$\ln(B(x))$	$B(x) > 0$
Logarithm in denominator	$\frac{1}{\ln(B(x))}$	$B(x) > 0$ and $B(x) \neq 1$
Rational under radical	$\sqrt{\frac{P(x)}{Q(x)}}$	$\frac{P(x)}{Q(x)} \geq 0$ and $Q(x) \neq 0$
Logarithm of a rational	$\ln\left(\frac{P(x)}{Q(x)}\right)$	$\frac{P(x)}{Q(x)} > 0$ and $Q(x) \neq 0$

Table 1.2: Fundamental algebraic conditions for determining domains of definition.

Worked Example: Domain of definition 01

Find the domain of definition for the function:

$$f(x) = \frac{\sqrt{4 - x^2}}{x - 1}$$

Solution: The expression is defined if and only if two conditions hold simultaneously:

1. The quantity inside the square root must be non-negative: $4 - x^2 \geq 0 \iff x^2 \leq 4 \iff x \in [-2, 2]$.
2. The denominator must not be zero: $x - 1 \neq 0 \iff x \neq 1$.

Combining both conditions, the domain of f is:

$$D_f = [-2, 1) \cup (1, 2].$$

Worked Example: Domain of definition 02

Determine the domain of definition of:

$$f_2(x) = \ln\left(\frac{2x - 4}{5 - x}\right)$$

Solution: The natural logarithm $\ln(u)$ is defined if and only if its argument is strictly positive:

$$\frac{2x - 4}{5 - x} > 0 \quad \text{with} \quad 5 - x \neq 0.$$

We locate the critical zeros of numerator and denominator:

- Numerator zero: $2x - 4 = 0 \iff x = 2$.
- Denominator zero: $5 - x = 0 \iff x = 5$ (forbidden vertical boundary).

We establish the sign chart across the three real intervals:

x	$-\infty$		2		5		$+\infty$
$2x - 4$	$-$		0	$+$		$+$	
$5 - x$	$+$		$+$	0		$-$	
$\frac{2x-4}{5-x}$	$-$		0	$+$	$\|\|$	$-$	

The fraction is strictly positive (> 0) precisely when $x \in (2, 5)$. Both endpoints are excluded because $x = 2$ gives $\ln(0)$ (undefined) and $x = 5$ gives division by zero.
Conclusion:

$$D_{f_2} = (2, 5).$$

1.2 Limits of a Function

The **limit** of a function describes the value that $f(x)$ approaches as the independent variable x approaches a given number a.

Definition: Formal Definition of a Limit (ε–δ)

Let $f(x)$ be defined on an open interval around a, except possibly at a itself. We say that $f(x)$ tends to a limit L as x tends to a, and we write:

$$\lim_{x \to a} f(x) = L$$

if for every $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that:

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

1.2.1 One-Sided Limits (Left and Right Limits)

Definition: Left and Right Limits

- The **right-hand limit** of f at a is denoted:

$$\lim_{x \to a^+} f(x) \quad (\text{where } x > a).$$

- The **left-hand limit** of f at a is denoted:

$$\lim_{x \to a^-} f(x) \quad (\text{where } x < a).$$

Criterion of Existence: The two-sided limit $\lim_{x \to a} f(x)$ exists and equals L if and only if both one-sided limits exist and are equal:

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L.$$

Worked Example: Evaluating Left and Right Limits

Let $f(x)$ be defined piecewise by:

$$f(x) = \begin{cases} x^2, & x < 2, \\ 3x - 2, & x \geq 2. \end{cases}$$

Determine whether $\lim_{x \to 2} f(x)$ exists.
Solution: We compute both one-sided limits at $x = 2$:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2) = (2)^2 = 4,$$
$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x - 2) = 3(2) - 2 = 6 - 2 = 4.$$

Since the left and right limits are equal:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 4,$$

we conclude that:

$$\lim_{x \to 2} f(x) = 4.$$

1.2.2 L'Hôpital's Rule

Theorem: L'Hôpital's Rule (Bernoulli–L'Hôpital)

Let f and g be differentiable on an open interval I containing a (except possibly at a), and suppose $g'(x) \neq 0$ for all $x \in I \setminus \{a\}$. Suppose that:

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0, \qquad \text{or} \qquad \lim_{x\to a} |f(x)| = \infty \quad \text{and} \quad \lim_{x\to a} |g(x)| = \infty.$$

If the derivative limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists in $\mathbb{R}$ or equals $\pm\infty$, then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Note: Here a can be a finite real number, a^+, a^-, $+\infty$, or $-\infty$.

Worked Example: Case 1: Indeterminate Form $\left[\frac{0}{0}\right]$ with Successive Application

Evaluate the limit:

$$L_1 = \lim_{x\to 0} \frac{\sin(3x) - 3x}{x^3}.$$

Step 1: Verify the form
Direct substitution: $\sin(0) - 0 = 0$ and $0^3 = 0$. This is an indeterminate form of type $\left[\frac{0}{0}\right]$.
Step 2: First application of L'Hôpital's Rule

$$L_1 = \lim_{x\to 0} \frac{\frac{d}{dx}[\sin(3x) - 3x]}{\frac{d}{dx}[x^3]} = \lim_{x\to 0} \frac{3\cos(3x) - 3}{3x^2} = \lim_{x\to 0} \frac{\cos(3x) - 1}{x^2}.$$

Evaluating at $x = 0$ gives $\frac{\cos(0)-1}{0^2} = \frac{0}{0}$, which is still indeterminate.
Step 3: Second application of L'Hôpital's Rule

$$L_1 = \lim_{x\to 0} \frac{\frac{d}{dx}[\cos(3x) - 1]}{\frac{d}{dx}[x^2]} = \lim_{x\to 0} \frac{-3\sin(3x)}{2x}.$$

Evaluating gives $\frac{0}{0}$, so we apply L'Hôpital a third time (or use $\lim_{x\to 0} \frac{\sin(3x)}{3x} = 1$):
Step 4: Third application of L'Hôpital's Rule

$$L_1 = \lim_{x\to 0} \frac{-3 \cdot 3\cos(3x)}{2} = \frac{-9\cos(0)}{2} = -\frac{9}{2}.$$

Conclusion:

$$\lim_{x\to 0} \frac{\sin(3x) - 3x}{x^3} = -\frac{9}{2}.$$

Worked Example: Case 3: Indeterminate Form $[0 \cdot \infty]$ (Reciprocal Inversion)

Evaluate the limit:

$$L_3 = \lim_{x\to 0^+} x\ln(x).$$

Step 1: Recognize the form
As $x \to 0^+$, $x \to 0$ and $\ln(x) \to -\infty$. This is the indeterminate form $[0 \times (-\infty)]$.
Step 2: Rewrite as a quotient $\left[\frac{\infty}{\infty}\right]$

We invert the polynomial factor to place it into the denominator:

$$x \ln(x) = \frac{\ln(x)}{\frac{1}{x}} = \frac{\ln(x)}{x^{-1}}.$$

Now as $x \to 0^+$, $\ln(x) \to -\infty$ and $x^{-1} \to +\infty$, giving the valid form $\left[\frac{-\infty}{+\infty}\right]$.

Step 3: Apply L'Hôpital's Rule

$$L_3 = \lim_{x \to 0^+} \frac{\frac{d}{dx}[\ln x]}{\frac{d}{dx}[x^{-1}]} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-x^{-2}} = \lim_{x \to 0^+} \left(\frac{1}{x} \cdot \left(-x^2\right)\right).$$

Step 4: Simplify

$$L_3 = \lim_{x \to 0^+} (-x) = 0.$$

Conclusion:

$$\lim_{x \to 0^+} x \ln(x) = 0.$$

1.3 Continuous Functions

Definition: Continuity at a Point

A function f is said to be **continuous at a point** a if the following three conditions are met:

1. $f(a)$ is defined (i.e., $a \in D_f$).
2. The limit $\lim_{x\to a} f(x)$ exists.
3. The limit equals the function value:

$$\lim_{x\to a} f(x) = f(a).$$

That is, the limit of $f(x)$ as x approaches a is equal to the actual value of f at that point.

Definition: Continuity on an Interval

A function f is continuous on an open interval (a, b) if it is continuous at every point $c \in (a, b)$.
A function f is continuous on a closed interval $[a, b]$ if it is continuous on (a, b), right-continuous at a ($\lim_{x\to a^+} f(x) = f(a)$), and left-continuous at b ($\lim_{x\to b^-} f(x) = f(b)$).

1.3.1 Types of Discontinuity

Definition: Classification of Discontinuities

When a function fails to be continuous at a point $x = a$, the discontinuity is classified as follows:

- When both one-sided limits $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and are finite, but they are not equal:

$$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x).$$

- When one or both of the one-sided limits tends to $+\infty$ or $-\infty$ as $x \to a$.

Worked Example:

1. Consider:

$$f(x) = \begin{cases} 2x + 1, & x < 1, \\ x + 3, & x \geq 1. \end{cases}$$

Left limit: $\lim_{x\to 1^-} f(x) = 2(1) + 1 = 3$.
Right limit: $\lim_{x\to 1^+} f(x) = 1 + 3 = 4$.
Since $3 \neq 4$, the function possesses a discontinuity at $x = 1$.

2. Consider:

$$f(x) = \frac{1}{x} \quad \text{at } x = 0.$$

We have $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$. Hence, f has a discontinuity at $x = 0$.

1.4 Differentiability of Functions of One Variable

Definition: Differentiability at a Point

Let f be a function defined in a neighborhood of x_0. We say that f is **differentiable** at x_0 if the limit of the difference quotient exists and is finite:

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}.$$

Notations: The derivative at x_0 can be designated by:

$$f'(x_0), \quad Df(x_0), \quad \frac{df}{dx}(x_0).$$

1.4.1 One-Sided Derivatives (Left and Right Derivatives)

Definition: Left and Right Derivatives

Let f be a function defined on an interval containing the point a:

- The **right derivative** of f at a is: $f'_+(a) = \lim_{x\to x_0^+} \frac{f(x)-f(x_0)}{x-x_0}$.
- The **left derivative** of f at a is: $f'_-(a) = \lim_{x\to x_0^-} \frac{f(x)-f(x_0)}{x-x_0}$.

Criterion of Differentiability: A function f is differentiable at a iff $f'_+(a) = f'_-(a)$.

Worked Example:

1. Consider:

$$f(x) = \begin{cases} 2x+1, & x < 1, \\ 3x, & x \geq 1. \end{cases}$$

Study of Continuity in Three Steps:

- **On** $]-\infty, 1[$**:** $f(x) = 2x+1$ is a polynomial function. It is defined and continuous on $\mathbb{R}$, hence f is continuous on $]-\infty, 1[$.
- **On** $]1, +\infty[$**:** $f(x) = 3x$ is a linear polynomial function. It is defined and continuous on $\mathbb{R}$, hence f is continuous on $]1, +\infty[$.
- **At** $x = 1$**:** We have $f(1) = 3(1) = 3$.
  Left limit: $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(2x+1) = 2(1)+1 = 3$.
  Right limit: $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(3x) = 3(1) = 3$.
  Since $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) = 3$, f is continuous at $x = 1$.

Conclusion: f is continuous on all of $\mathbb{R}$.

2. Study of Differentiability (Derivability) in Three Steps:

- **On** $]-\infty, 1[$**:** $f(x) = 2x+1$ is a polynomial function. It is defined and differentiable on $\mathbb{R}$, hence f is differentiable on $]-\infty, 1[$, with $f'(x) = 2$.
- **On** $]1, +\infty[$**:** $f(x) = 3x$ is a linear polynomial function. It is defined and differentiable on $\mathbb{R}$, hence f is differentiable on $]1, +\infty[$, with $f'(x) = 3$.

- **At $x = 1$:** We evaluate the one-sided difference quotients:

$$f'_-(1) = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{(2x+1)-3}{x-1} = \lim_{x\to 1^-} \frac{2(x-1)}{x-1} = 2.$$

$$f'_+(1) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{3x-3}{x-1} = \lim_{x\to 1^+} \frac{3(x-1)}{x-1} = 3.$$

Since $f'_-(1) = 2 \neq 3 = f'_+(1)$, the function f is not differentiable at $x = 1$.

Conclusion: f is differentiable on $]-\infty, 1[\cup]1, +\infty[= \mathbb{R} \setminus \{1\}$.

1.4.2 Geometrical Interpretation and Tangent Line

Definition: Tangent Line to a Curve

If f is differentiable at a, the derivative $f'(a)$ represents the **slope of the tangent line** to the curve $y = f(x)$ at the point $(a, f(a))$. The equation of the tangent line is given by:

$$y = f(a) + f'(a)(x - a).$$

Theorem: Differentiability Implies Continuity

If a function f is differentiable at a point x_0, then f is continuous at x_0. (The converse is false: $f(x) = |x|$ is continuous at 0, but not differentiable at 0).

1.4.3 Differentiation Formulas

Rule	**Formula**	**Example**
Constant	$(c)' = 0$	$(3)' = 0$
Power	$(x^n)' = nx^{n-1}$	$(x^4)' = 4x^3$
Sum	$(f+g)' = f' + g'$	$(x^3 + x^2 + 3x)' = 3x^2 + 2x + 3$
Product	$(fg)' = f'g + fg'$	$(x^2 \sin x)' = 2x \sin x + x^2 \cos x$
Quotient	$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$	$\left(\frac{x^2}{\sin x}\right)' = \frac{2x \sin x - x^2 \cos x}{\sin^2 x}$
Chain Rule	$[f(g(x))]' = f'(g(x)) \cdot g'(x)$	$[\sin(x^2)]' = \cos(x^2) \cdot 2x = 2x \cos(x^2)$

Table 1.3: Fundamental rules of differentiation with examples.

1.5 Integration and Primitives

Integration is one of the two pillars of calculus. While differentiation determines instantaneous rates of change, integration computes accumulated quantities and total area under a curve.

Definition: The Indefinite Integral

A **primitive** (or antiderivative) of a function $f(x)$ is a differentiable function $F(x)$ such that $F'(x) = f(x)$. The indefinite integral represents the family of all primitives:

$$\int f(x)\,dx = F(x) + c, \qquad c \in \mathbb{R}.$$

1.5.1 Properties of the Indefinite Integral

1. **Scalar Multiplication:** $\int af(x)\,dx = a\int f(x)\,dx = aF(x) + c.$
2. **Sum Rule:** $\int \big(f(x) \pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$
3. **Affine Scaling Property:** $\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + c$ $(a \neq 0)$.

Worked Example: Linearity and Term-by-Term Integration

Evaluate:

$$I = \int \left(4x^3 - \frac{3}{x} + 5e^x - 2\sin(x)\right) dx.$$

Solution: Using scalar multiplication and the sum rule, we integrate term-by-term:

$$I = 4\int x^3\,dx - 3\int \frac{1}{x}\,dx + 5\int e^x\,dx - 2\int \sin(x)\,dx.$$

Applying standard primitives:

$$I = 4\left(\frac{x^4}{4}\right) - 3\ln|x| + 5e^x - 2(-\cos(x)) + c = x^4 - 3\ln|x| + 5e^x + 2\cos(x) + c, \quad c \in \mathbb{R}.$$

Worked Example: Application of the Affine Scaling Property

Evaluate:

$$I_1 = \int \cos(3x-4)\,dx \qquad \text{and} \qquad I_2 = \int e^{5x+2}\,dx.$$

Solution: Recall property 3: $\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + c.$

1. For I_1: $f(t) = \cos(t) \implies F(t) = \sin(t)$, with $a = 3$ and $b = -4$:

$$I_1 = \frac{1}{3}\sin(3x-4) + c_1, \quad c_1 \in \mathbb{R}.$$

2. For I_2: $f(t) = e^t \implies F(t) = e^t$, with $a = 5$ and $b = 2$:

$$I_2 = \frac{1}{5}e^{5x+2} + c_2, \quad c_2 \in \mathbb{R}.$$

1.5.2 The 26 Usual Primitives

#	**Function** $f(x)$	**Primitive** $F(x)+c$ $(c\in\mathbb{R})$
1	$x^n\ (n\neq -1)$	$\frac{1}{n+1}x^{n+1}+c$
2	$x^{-1}=\frac{1}{x}$	$\ln(\lvert x\rvert)+c$
3	e^x	e^x+c
4	$a^x\ (a>0, a\neq 1)$	$\frac{a^x}{\ln a}+c$
5	$\sin(x)$	$-\cos(x)+c$
6	$\cos(x)$	$\sin(x)+c$
7	$\frac{1}{\sin^2(x)}$	$-\cot(x)+c$
8	$\frac{1}{\cos^2(x)}$	$\tan(x)+c$
9	$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x)+c=-\arccos(x)+c$
10	$\frac{1}{1+x^2}$	$\arctan(x)+c$
11	$\frac{1}{\sqrt{a^2-x^2}}\ (a>0)$	$\arcsin\left(\frac{x}{a}\right)+c$
12	$\frac{1}{a^2+x^2}\ (a\neq 0)$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)+c$
13	$\frac{1}{a^2-x^2}\ (a\neq 0)$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert+c$
14	$\frac{1}{\sqrt{x^2\pm a^2}}$	$\ln\left\lvert x+\sqrt{x^2\pm a^2}\right\rvert+c$
15	$\ln(x)$	$x\ln(x)-x+c$
16	$\log_a(x)$	$x\log_a(x)-x\log_a(e)+c$
17	$\tan(x)$	$-\ln(\lvert\cos(x)\rvert)+c$
18	$\sinh(x)$	$\cosh(x)+c$
19	$\cosh(x)$	$\sinh(x)+c$
20	$\tanh(x)$	$\ln(\cosh(x))+c$
21	$\frac{u'(x)}{u(x)}$	$\ln(\lvert u(x)\rvert)+c$
22	$u'(x)e^{u(x)}$	$e^{u(x)}+c$
23	$\frac{u'(x)}{\sqrt{u(x)}}$	$2\sqrt{u(x)}+c$
24	$u'(x)[u(x)]^n\ (n\neq -1)$	$\frac{1}{n+1}[u(x)]^{n+1}+c$
25	$u'(x)\sin(u(x))$	$-\cos(u(x))+c$
26	$u'(x)\cos(u(x))$	$\sin(u(x))+c$

Table 1.4: Table of standard primitives and composite rules.

Worked Example: Inverse Trigonometric and Rational Primitives

Evaluate using the 26 Usual Primitives:

$$I_a=\int\frac{dx}{\sqrt{9-x^2}},\qquad I_b=\int\frac{dx}{4+x^2},\qquad I_c=\int\frac{dx}{16-x^2}.$$

Solution:

1. For I_a: Matches **Rule 11** $\int\frac{dx}{\sqrt{a^2-x^2}}=\arcsin\left(\frac{x}{a}\right)+c$ with $a=3$:

$$I_a=\arcsin\left(\frac{x}{3}\right)+c_1.$$

2. For I_b: Matches **Rule 12** $\int\frac{dx}{a^2+x^2}=\frac{1}{a}\arctan\left(\frac{x}{a}\right)+c$ with $a=2$:

$$I_b=\frac{1}{2}\arctan\left(\frac{x}{2}\right)+c_2.$$

3. For I_c: Matches **Rule 13** $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$ with $a = 4$:

$$I_c = \frac{1}{2(4)}\ln\left|\frac{4+x}{4-x}\right| + c_3 = \frac{1}{8}\ln\left|\frac{4+x}{4-x}\right| + c_3.$$

Worked Example: Composite Rules: u'/u and $u' \cdot u^n$

Evaluate using composite rules:

$$I_1 = \int \frac{3x^2+2x}{x^3+x^2+5}\,dx \qquad \text{and} \qquad I_2 = \int \cos(x)\sin^5(x)\,dx.$$

Solution:

1. For I_1: Let $u(x) = x^3 + x^2 + 5$. Then $u'(x) = 3x^2 + 2x$. Using **Rule 21** $\int \frac{u'(x)}{u(x)}\,dx = \ln|u(x)| + c$:
$$I_1 = \ln\left|x^3 + x^2 + 5\right| + c_1.$$

2. For I_2: Let $u(x) = \sin(x)$ with $n = 5$. Then $u'(x) = \cos(x)$. Using **Rule 24** $\int u'(x)[u(x)]^n\,dx = \frac{1}{n+1}[u(x)]^{n+1} + c$:

$$I_2 = \frac{1}{5+1}[\sin(x)]^{5+1} + c_2 = \frac{1}{6}\sin^6(x) + c_2.$$

1.6 Integration Methods

1.6.1 Integration by Substitution (Change of Variables)

Setting $u = h(x)$, we have $du = h'(x)\,dx$:

$$\int f(h(x))h'(x)\,dx = \int f(u)\,du = F(u) + c = F(h(x)) + c.$$

Worked Example: Integration by Substitution

Evaluate $\int 2xe^{x^2}\,dx$.
Solution: Let $u = x^2 \implies du = 2x\,dx$.

$$\int e^u\,du = e^u + c = e^{x^2} + c.$$

Worked Example: Algebraic Substitution with Radicals

Evaluate:

$$I = \int x\sqrt{2x+1}\,dx.$$

Solution: Let $u = 2x + 1$. Then $x = \frac{u-1}{2}$ and $dx = \frac{du}{2}$. Substituting into I:

$$I = \int \left(\frac{u-1}{2}\right) u^{1/2} \left(\frac{du}{2}\right) = \frac{1}{4}\int \left(u^{3/2} - u^{1/2}\right) du$$
$$= \frac{1}{4}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right] + c = \frac{1}{10}u^{5/2} - \frac{1}{6}u^{3/2} + c.$$

Factoring the common factor $\frac{1}{30}u^{3/2}(3u-5)$ and replacing $u = 2x+1$:

$$I = \frac{1}{30}(2x+1)^{3/2}\big(3(2x+1) - 5\big) + c = \frac{1}{15}(3x-1)(2x+1)^{3/2} + c, \quad c \in \mathbb{R}.$$

Worked Example: Transcendental Substitution: Logarithm and Trigonometry

Evaluate:

$$I_1 = \int \frac{\ln^3(x)}{x}\,dx \qquad \text{and} \qquad I_2 = \int \frac{\sin(x)}{1+\cos^2(x)}\,dx.$$

Solution:

1. For I_1: Let $u = \ln(x) \implies du = \frac{1}{x}\,dx$:

$$I_1 = \int u^3\,du = \frac{1}{4}u^4 + c_1 = \frac{1}{4}\ln^4(x) + c_1.$$

2. For I_2: Let $u = \cos(x) \implies du = -\sin(x)\,dx$. The integral becomes:

$$I_2 = \int \frac{-du}{1+u^2} = -\arctan(u) + c_2 = -\arctan(\cos(x)) + c_2, \quad c_2 \in \mathbb{R}.$$

1.6.2 Integration by Parts

Theorem: Integration by Parts Formula

Let $u(x)$ and $v(x)$ be continuously differentiable functions:

$$\int u'(x)v(x)\,dx = u(x)v(x) - \int u(x)v'(x)\,dx.$$

Worked Example: Integration by Parts Example

Evaluate $I = \int xe^x\,dx$.
Solution: Let $v(x) = x \implies v'(x) = 1$ and $u'(x) = e^x \implies u(x) = e^x$.

$$\int xe^x\,dx = xe^x - \int (1)e^x\,dx = (x-1)e^x + c.$$

Worked Example: Repeated Integration by Parts

Evaluate:

$$I = \int x^2 \sin(x)\,dx.$$

Solution:
First application: Let $v(x) = x^2 \implies v'(x) = 2x$, and $u'(x) = \sin(x) \implies u(x) = -\cos(x)$.

$$I = -x^2\cos(x) - \int (-2x\cos(x))\,dx = -x^2\cos(x) + 2\int x\cos(x)\,dx.$$

Second application: For $J = \int x\cos(x)\,dx$, choose:

$$v_1(x) = x \implies v_1'(x) = 1, \qquad u_1'(x) = \cos(x) \implies u_1(x) = \sin(x).$$

$$J = x\sin(x) - \int \sin(x)\,dx = x\sin(x) - (-\cos(x)) = x\sin(x) + \cos(x).$$

Substituting J into I:

$$I = -x^2\cos(x) + 2\big(x\sin(x) + \cos(x)\big) + c = -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + c, \quad c \in \mathbb{R}.$$

Worked Example: The Invisible Factor Technique: $\int \ln(x)\,dx$ and $\int \arctan(x)\,dx$

Evaluate:

$$I_1 = \int \ln(x)\,dx \qquad \text{and} \qquad I_2 = \int \arctan(x)\,dx.$$

Solution: When integrating isolated inverse or logarithmic functions, we introduce the factor $u'(x) = 1 \implies u(x) = x$.

1. For I_1: Let $v(x) = \ln(x) \implies v'(x) = \frac{1}{x}$, and $u'(x) = 1 \implies u(x) = x$:

$$I_1 = x\ln(x) - \int x\cdot\frac{1}{x}\,dx = x\ln(x) - \int 1\,dx = x\ln(x) - x + c_1.$$

2. For I_2: Let $v(x) = \arctan(x) \implies v'(x) = \frac{1}{1+x^2}$, and $u'(x) = 1 \implies u(x) = x$:

$$I_2 = x\arctan(x) - \int \frac{x}{1+x^2}\,dx.$$

Recognizing $\frac{x}{1+x^2} = \frac{1}{2}\frac{(1+x^2)'}{1+x^2}$, we obtain:

$$I_2 = x\arctan(x) - \frac{1}{2}\ln(1+x^2) + c_2, \quad c_2 \in \mathbb{R}.$$

Worked Example: Cyclic Integration by Parts

Evaluate:

$$I = \int e^{2x}\cos(3x)\,dx.$$

Solution:

Step 1 (First parts): Let $v(x) = \cos(3x) \implies v'(x) = -3\sin(3x)$, and $u'(x) = e^{2x} \implies u(x) = \frac{1}{2}e^{2x}$.

$$I = \frac{1}{2}e^{2x}\cos(3x) + \frac{3}{2}\int e^{2x}\sin(3x)\,dx.$$

Step 2 (Second parts): On the remaining integral, set $v_1(x) = \sin(3x) \implies v_1'(x) = 3\cos(3x)$, and $u_1'(x) = e^{2x} \implies u_1(x) = \frac{1}{2}e^{2x}$:

$$\int e^{2x}\sin(3x)\,dx = \frac{1}{2}e^{2x}\sin(3x) - \frac{3}{2}\underbrace{\int e^{2x}\cos(3x)\,dx}_{I}.$$

Step 3 (Recombination): Substituting back:

$$I = \frac{1}{2}e^{2x}\cos(3x) + \frac{3}{4}e^{2x}\sin(3x) - \frac{9}{4}I.$$

Step 4 (Solve for I): Adding $\frac{9}{4}I$ to both sides:

$$\frac{13}{4}I = \frac{e^{2x}}{4}\big(2\cos(3x) + 3\sin(3x)\big) \implies I = \frac{e^{2x}}{13}\big(2\cos(3x) + 3\sin(3x)\big) + c, \quad c \in \mathbb{R}.$$

1.7 Integration of Rational Functions

Worked Example: Partial Fractions Decomposition with Irreducible Quadratic

Evaluate:

$$I_1 = \int \frac{1}{(x^2+1)(x-1)}\,dx.$$

Partial fraction setup:

$$\frac{1}{(x^2+1)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

Multiplying by $(x^2+1)(x-1)$ and solving: $A = \frac{1}{2}, B = -\frac{1}{2}, C = -\frac{1}{2}$.

$$\begin{aligned} I_1 &= \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{x+1}{x^2+1}\,dx \\ &= \frac{1}{2}\ln|x-1| - \frac{4}{1}\ln(x^2+1) - \frac{1}{2}\arctan(x) + c, \quad c \in \mathbb{R}. \end{aligned}$$

Worked Example: Case 1: Distinct Linear Factors

Evaluate:

$$I = \int \frac{5x-2}{x^2-x-2}\,dx.$$

Solution: The denominator factors as $x^2 - x - 2 = (x-2)(x+1)$. The partial fraction decomposition is:

$$\frac{5x-2}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}.$$

Using the Heaviside cover-up method:

$$A = \left.\frac{5x-2}{x+1}\right|_{x=2} = \frac{10-2}{3} = \frac{8}{3}, \qquad B = \left.\frac{5x-2}{x-2}\right|_{x=-1} = \frac{-5-2}{-3} = \frac{7}{3}.$$

Integrating directly:

$$I = \frac{8}{3}\int \frac{dx}{x-2} + \frac{7}{3}\int \frac{dx}{x+1} = \frac{8}{3}\ln|x-2| + \frac{7}{3}\ln|x+1| + c, \quad c \in \mathbb{R}.$$

Worked Example: Case 2: Repeated Linear Factor

Evaluate:

$$I = \int \frac{3x+5}{(x-1)^2}\,dx.$$

Solution: For a repeated factor $(x-1)^2$, the partial fraction form is:

$$\frac{3x+5}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}.$$

Clearing denominators: $3x + 5 = A(x-1) + B = Ax + (B - A)$. Matching coefficients:

$$A = 3, \qquad B - A = 5 \implies B = 5 + 3 = 8.$$

Integrating term-by-term:

$$I = \int \frac{3}{x-1}\,dx + \int \frac{8}{(x-1)^2}\,dx = 3\ln|x-1| - \frac{8}{x-1} + c, \quad c \in \mathbb{R}.$$

Worked Example: Case 3: Improper Rational Function (Polynomial Long Division)

Evaluate:

$$I = \int \frac{x^3 + 2}{x^2 - 1}\, dx.$$

Solution: Since deg(numerator) $= 3 \geq$ deg(denominator) $= 2$, we first perform polynomial long division:

$$x^3 + 2 = x(x^2 - 1) + (x + 2) \implies \frac{x^3 + 2}{x^2 - 1} = x + \frac{x + 2}{x^2 - 1}.$$

Now decompose the proper remainder $\frac{x+2}{(x-1)(x+1)}$:

$$\frac{x + 2}{(x - 1)(x + 1)} = \frac{A}{x - 1} + \frac{B}{x + 1}.$$

Using the cover-up method:

$$A = \left.\frac{x + 2}{x + 1}\right|_{x=1} = \frac{3}{2}, \qquad B = \left.\frac{x + 2}{x - 1}\right|_{x=-1} = -\frac{1}{2}.$$

Integrating all components:

$$\begin{aligned} I &= \int x\, dx + \frac{3}{2}\int \frac{dx}{x - 1} - \frac{1}{2}\int \frac{dx}{x + 1} \\ &= \frac{1}{2}x^2 + \frac{3}{2}\ln|x - 1| - \frac{1}{2}\ln|x + 1| + c, \quad c \in \mathbb{R}. \end{aligned}$$

Worked Example: Case 4: Irreducible Quadratic via Completing the Square

Evaluate:

$$I = \int \frac{2x + 7}{x^2 + 4x + 13}\, dx.$$

Solution: The denominator has discriminant $\Delta = 4^2 - 4(13) = -36 < 0$ (irreducible).
Step 1 (Derivative splitting): The derivative of the denominator is $(x^2 + 4x + 13)' = 2x + 4$. Rewrite the numerator:

$$2x + 7 = (2x + 4) + 3.$$

Thus:

$$I = \int \frac{2x + 4}{x^2 + 4x + 13}\, dx + 3\int \frac{dx}{x^2 + 4x + 13}.$$

Step 2 (Evaluate logarithmic part): By Rule 21 (u'/u):

$$\int \frac{2x + 4}{x^2 + 4x + 13}\, dx = \ln(x^2 + 4x + 13).$$

(Absolute value is omitted since $x^2 + 4x + 13 > 0$ for all $x \in \mathbb{R}$).
Step 3 (Complete the square for arctan part):

$$x^2 + 4x + 13 = (x + 2)^2 + 9 = (x + 2)^2 + 3^2.$$

Using Rule 12 with affine scaling $(a = 3)$:

$$3\int \frac{dx}{(x + 2)^2 + 3^2} = 3 \cdot \frac{1}{3}\arctan\left(\frac{x + 2}{3}\right) = \arctan\left(\frac{x + 2}{3}\right).$$

Conclusion:

$$I = \ln(x^2 + 4x + 13) + \arctan\left(\frac{x+2}{3}\right) + c, \quad c \in \mathbb{R}.$$