Chapter 1

Real Functions

1.1 Numerical Functions

Definition 1.1

Let E and F be two sets and f be a relation from the set E to the set F. We say that f is a function if every element of E is associated with at most one element of F, and we write:

$$\begin{array}{rccc} f: & E & \longrightarrow & F \\ & x & \longmapsto & f(x) = y \end{array}$$

Definition 1.2

We say that f is a numerical function if and only if:

$$\begin{array}{rccc} f: & E \subset \mathbb{R} & \longrightarrow & F \subset \mathbb{R} \\ & x & \longmapsto & f(x) = y \end{array}$$

In other words, f is a numerical function if and only if for every element x in E, its image in F is at most one real number.

Example 1.1 (Linear Function)

The simplest numerical function is the linear function:

$$\begin{array}{rccc} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & f(x) = ax + b \end{array}$$

where a and b are real constants. Its graph is a straight line with slope a and y-intercept b.

Example 1.2 (Inverse Function)

The inverse function:

$$\begin{array}{rcl} f : \mathbb{R} \setminus \{0\} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{1}{x} \end{array}$$

is defined for all real numbers except zero. Its graph is a hyperbola with two branches.

1.1.1 Domain of Definition

Definition 1.3

Let f be a numerical function. The domain of f, denoted by $\mathrm{Dom}(f)$, is the set of all real numbers x such that $f(x)$ is a well-defined real number:

$$\mathrm{Dom}(f) = \{x \in \mathbb{R} \mid f(x) \in \mathbb{R}\}$$

In other words, the domain of a numerical function is the set of all values for which the function is defined and has a real number output:

$$\begin{array}{rcl} f : \mathrm{Dom}(f) \subset \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) \end{array}$$

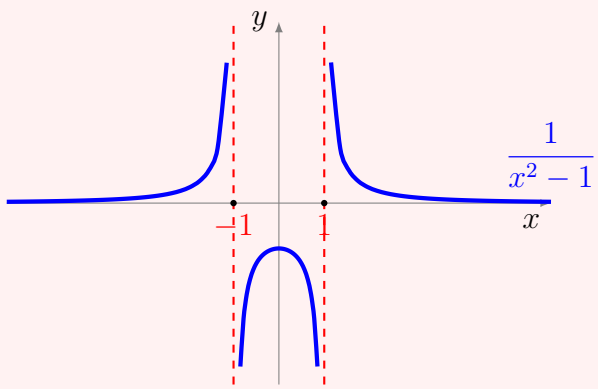
Example 1.3 (Rational Function)

Consider the function:

$$\begin{array}{rcl} f : \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = \dfrac{1}{(x^2 - 1)} \end{array}$$

The variable x is in the denominator of the function. Since division by zero is undefined, we must exclude values that make the denominator zero:

$$\begin{aligned} x^2 - 1 = 0 &\iff (x-1)(x+1) = 0 \\ &\iff x = 1 \text{ or } x = -1 \\ &\iff D_f = \mathbb{R} \setminus \{-1, 1\} \\ &\iff D_f =]-\infty, -1[\cup]-1, 1[\cup]1, +\infty[\end{aligned}$$



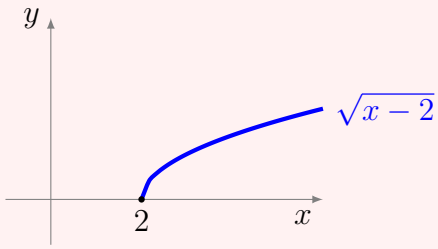
Example 1.4 (Square Root Function)

Consider the function:

$$\begin{array}{ccc} f : \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = \sqrt{x-2} \end{array}$$

The domain consists of all real numbers x for which the expression under the square root is non-negative:

$$\begin{aligned} x - 2 &\geq 0 \\ x &\geq 2 \\ D_f &= [2, +\infty[\end{aligned}$$



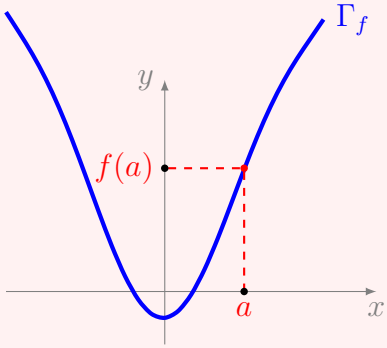
1.1.2 Function Curves and Their Properties

Definition 1.4

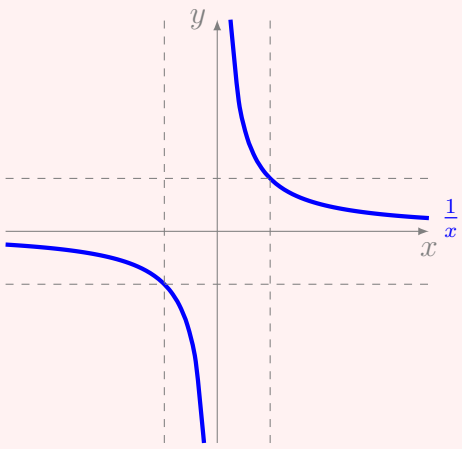
The graph of the function $f : U \to \mathbb{R}$ is the subset Γ_f of $\mathbb{R}^2$ defined as:

$$\Gamma_f = \big\{(x, f(x)) \mid x \in U\big\}.$$

Example 1.5 (Basic Function Graphs)



Graph of $f(x) = \frac{1}{2} + \frac{x^2}{2} + \sin\left(\frac{3(x-1)}{2}\right)$

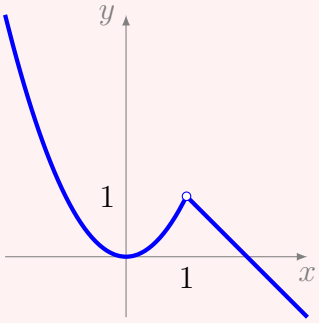


Graph of $f(x) = \frac{1}{x}$ with asymptotes

Example 1.6 (Piecewise Function)

Consider the piecewise function:

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 2 - x & \text{if } x > 1 \end{cases}$$



Graph of a piecewise function with transition at $x = 1$

1.2 Parity and Periodicity

1.2.1 Even and Odd Functions

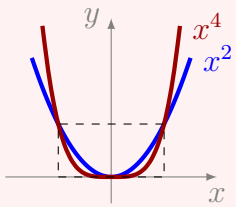
Definition 1.5

A function f is:

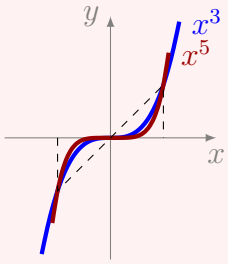
- **Even** if $\forall x \in D_f : f(-x) = f(x)$ (symmetric about y-axis)
- **Odd** if $\forall x \in D_f : f(-x) = -f(x)$ (symmetric about origin)

Example 1.7 (Common Even and Odd Functions)

Function Type	Example	Parity
Polynomial with even powers	$f(x) = x^4 - 2x^2 + 1$	Even
Polynomial with odd powers	$f(x) = x^3 - x$	Odd
Mixed polynomial	$f(x) = x^3 + x^2$	Neither
Trigonometric	$f(x) = \cos x$	Even
Trigonometric	$f(x) = \sin x$	Odd
Absolute value	$f(x) = \lvert x \rvert$	Even



Even functions: x^2 and x^4



Odd functions: x^3 and x^5

1.2.2 Periodic Functions

Definition 1.6

A function f is **periodic** with period $T > 0$ if:

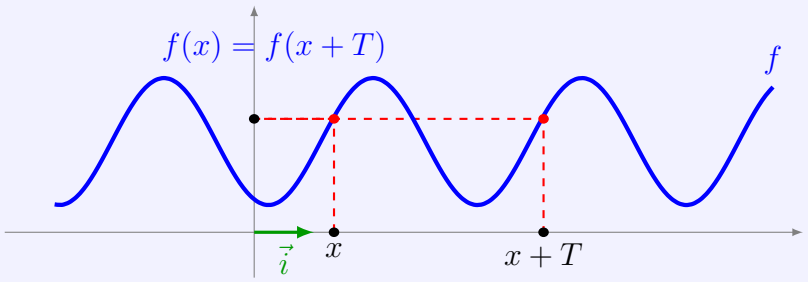
$$\forall x \in D_f : f(x + T) = f(x)$$

The smallest such T is called the fundamental period.

Definition 1.7

We say that f is a periodic function if there exists $k > 0$ where :

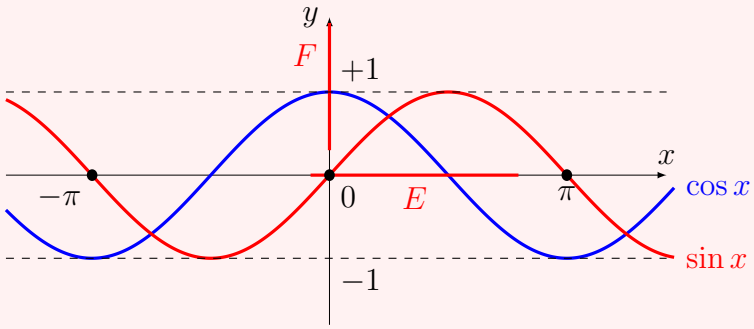
$$\forall x \in D_f : f(x+k) = f(x).$$



The part of the graph that corresponds to the smallest repeating pattern of a periodic function is called one cycle. The gap between two consecutive points that mark the end of the same cycle is called the period.

Example 1.8

The sine and cosine functions are periodic functions with a period of 2π, while the tangent function is a periodic function with a period of π.



1.2.3 Function Positivity and Negativity

Definition 1.8

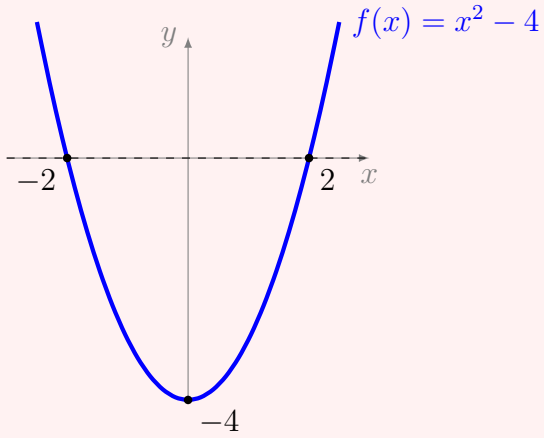
For $f : D_f \to \mathbb{R}$ and $\Delta \subseteq D_f$:

- f is **positive** on Δ if $\forall x \in \Delta : f(x) \geq 0$
- f is **strictly positive** on Δ if $\forall x \in \Delta : f(x) > 0$
- f is **negative** on Δ if $\forall x \in \Delta : f(x) \leq 0$
- f is **strictly negative** on Δ if $\forall x \in \Delta : f(x) < 0$

Example 1.9 (Sign Analysis)

Consider $f(x) = x^2 - 4$:

- Positive on $(-\infty, -2] \cup [2, \infty)$
- Negative on $(-2, 2)$
- Zero at $x = \pm 2$



Sign analysis of $f(x) = x^2 - 4$

Remark 1.2.1.

- *If the function f is positive, its graph lies above the x-axis, and conversely, if the function f is negative, its graph lies below the x-axis.*
- *If the function f is strictly positive or strictly negative, its graph never intersects the x-axis.*

1.2.4 Operations on functions

Let $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}$ be two defined functions on the same part U of the set $\mathbb{R}$. From this, we can define the following functions:

1) The sum of the functions f and g is the function $f + g : U \to \mathbb{R}$ defined as follows:

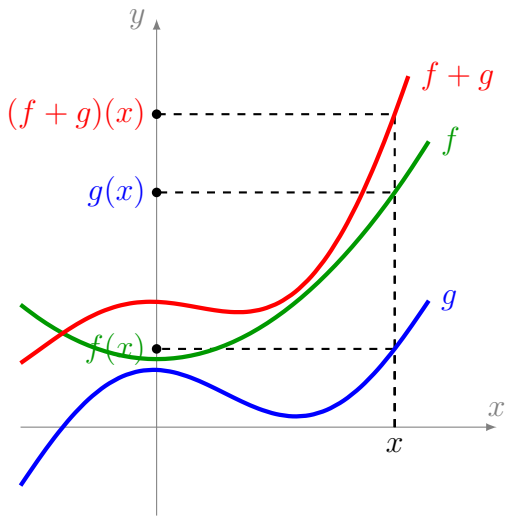
$$\forall x \in U, (f + g)(x) = f(x) + g(x).$$

2) The product of the functions f and g is the function $f \cdot g : U \to \mathbb{R}$ defined as follows:

$$\forall x \in U, (f \cdot g)(x) = f(x) \cdot g(x).$$

3) The product by scalar $\lambda \in \mathbb{R}$ and the function f is the function $\lambda \cdot f : U \to \mathbb{R}$ defined as follows:

$$\forall x \in U, (\lambda \cdot f)(x) = \lambda \cdot f(x).$$



1.2.5 Function Monotony

Let f be a function defined on its domain D_f, and let I be a subset of D_f.

Definition 1.9 (Increasing Function)

A function f is **increasing** on I if and only if:

$$\forall (x, y) \in I^2 : x > y \implies f(x) \geq f(y).$$

This means the function's output never decreases as the input increases. The function may have flat (constant) regions.

Example 1.10

The function $f(x) = \lfloor x \rfloor$ (floor function) is increasing on $\mathbb{R}$, though not strictly increasing. For example:

- At $x = 1.5$, $f(x) = 1$
- At $x = 2$, $f(x) = 2$
- At $x = 2.3$, $f(x) = 2$

We see that while the function never decreases, it remains constant between integer values.

Definition 1.10 (Strictly Increasing Function)

A function f is **strictly increasing** on I if and only if:

$$\forall (x, y) \in I^2 : x > y \implies f(x) > f(y).$$

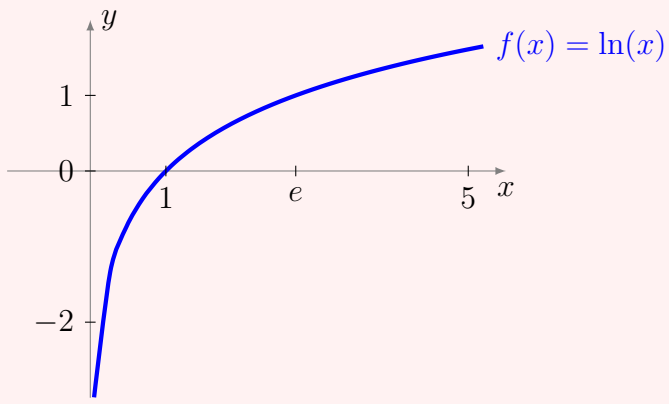
In this case, the function has no flat regions - it always increases as the input increases.

Example 1.11

The natural logarithm function $f(x) = \ln(x)$ is strictly increasing on its domain $]0, +\infty[$:

- $\ln(1) = 0$
- $\ln(e) \approx 1.718$
- $\ln(10) \approx 2.303$

As we can see, as x increases, $\ln(x)$ always increases.



Definition 1.11 (Decreasing Function)

A function f is **decreasing** on I if and only if:

$$\forall (x, y) \in I^2 : x > y \implies f(x) \leq f(y).$$

This means the function's output never increases as the input increases. The function may have flat regions.

Example 1.12

The function $f(x) = -|x|$ is decreasing on $\mathbb{R}$:

- $f(-2) = -2$

- $f(-1) = -1$
- $f(0) = 0$
- $f(1) = -1$
- $f(2) = -2$

Notice that between $x = -1$ and $x = 1$, the function increases and then decreases, but overall it's not monotonic on $\mathbb{R}$. However, it's decreasing on $[0, +\infty[$.

Definition 1.12 (Strictly Decreasing Function)

A function f is **strictly decreasing** on I if and only if:

$$\forall (x, y) \in I^2 : x > y \implies f(x) < f(y).$$

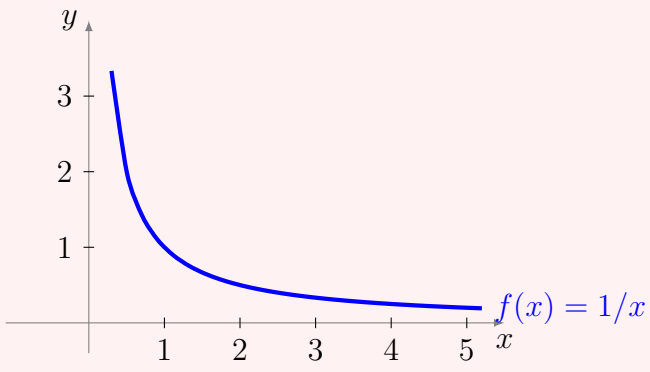
In this case, the function has no flat regions - it always decreases as the input increases.

Example 1.13

The reciprocal function $f(x) = \frac{1}{x}$ is strictly decreasing on $]0, +\infty[$:

- $f(1) = 1$
- $f(2) = 0.5$
- $f(10) = 0.1$

As x increases, $f(x)$ always decreases.



1.2.6 Finite function

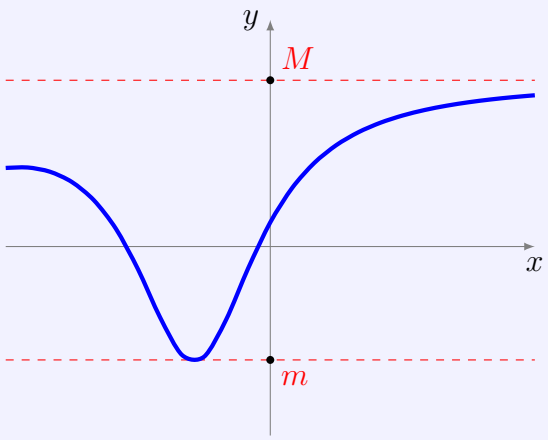
Definition 1.13

Let f be a numerical function defined on the set D_f

1) We say that f is bounded above if and only if there exists a real number M such that:
$$\forall x \in D_f : \quad f(x) \leq M.$$

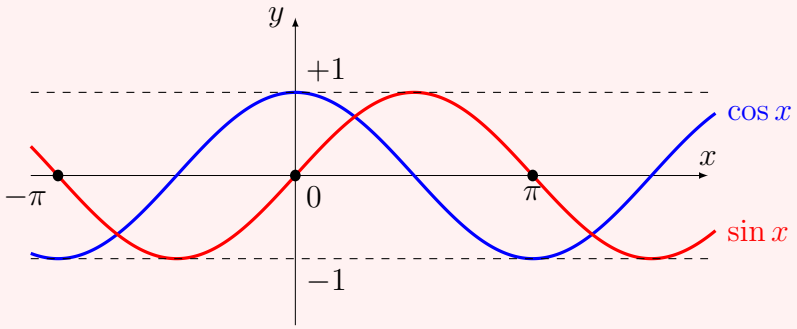
2) We say that f is bounded below if and only if there exists a real number m such that:
$$\forall x \in D_f : \quad m \leq f(x).$$

3) We say that f is bounded if and only if there exist two real numbers m and M such that:
$$\forall x \in D_f : \quad m \leq f(x) \leq M.$$



Example 1.14

The sine and cosine functions are bounded functions.



1.2.7 Max and min values of a function

Definition 1.14

Let f be a numerical function defined on the set D_f, and let $x_0 \in D_f$ and I be a subset of D_f.

1) We say that the number $f(x_0)$ is the absolute maximum value of the function f at the point x_0 if:
$$\forall x \in D_f : \quad f(x) \leq f(x_0).$$

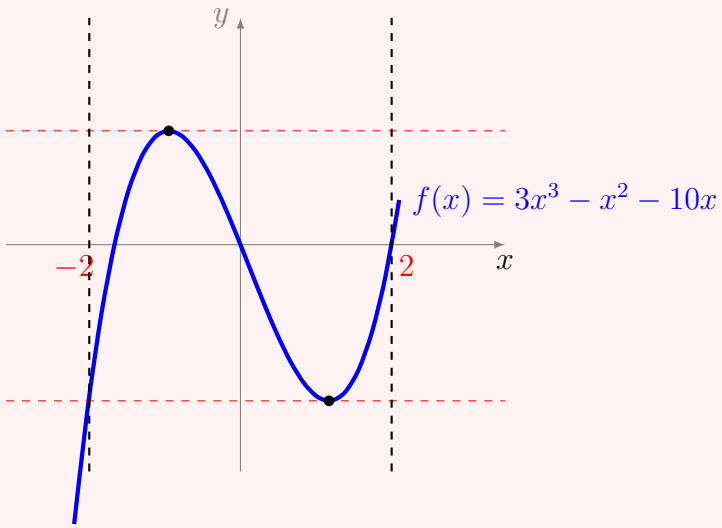
2) We say that the number $f(x_0)$ is a relative maximum value of the function f at the point x_0 in the domain I if $x_0 \in I$ and:
$$\forall x \in I \quad f(x) \leq f(x_0).$$

3) We say that the number $f(x_0)$ is the absolute minimum value of the function f at the point x_0 if:
$$\forall x \in D_f \quad f(x) \geq f(x_0).$$

4) We say that the number $f(x_0)$ is a relative minimum value of the function f at the point x_0 in the domain I if $x_0 \in I$ and:
$$\forall x \in I \quad f(x) \geq f(x_0).$$

Example 1.15

The function f has an upper limit and a lower limit at the two specified points in the graph on the domain $[2, 2]$.



1.3 Limits

1.3.1 Definitions and Fundamental Concepts

Neighborhoods and Point Limits

Definition 1.15 (Neighborhood)

A subset $V \subseteq \mathbb{R}$ is called a **neighborhood** of a point $x_0 \in \mathbb{R}$ if there exists an open interval (a, b) containing x_0 such that $(a, b) \subseteq V$. In other words, V contains all points sufficiently close to x_0.

Example 1.16

- The interval $(1.9, 2.1)$ is a neighborhood of $x_0 = 2$
- The set $[0, \infty)$ is a neighborhood of $x_0 = 1$ but not of $x_0 = 0$
- The entire real line $\mathbb{R}$ is a neighborhood of every point

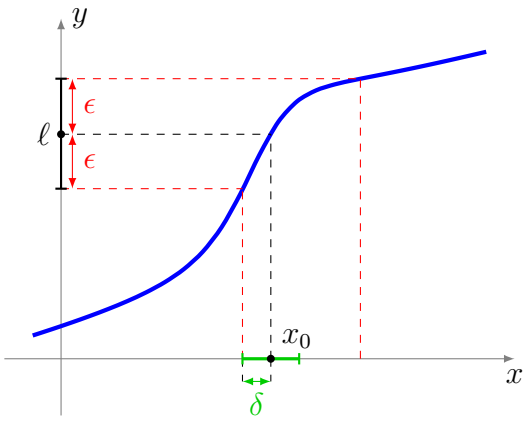
Definition 1.16 (Finite Limit at a Point)

Let $f : I \to \mathbb{R}$ be a function defined on some interval $I \subseteq \mathbb{R}$, and let $x_0 \in \mathbb{R}$ be a limit point of I. We say that f has **limit** $\ell \in \mathbb{R}$ as x approaches x_0 if:

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x \in I :\ 0 < |x - x_0| < \delta \implies |f(x) - \ell| < \epsilon.$$

We write this as:

$$\lim_{x \to x_0} f(x) = \ell \quad \text{or} \quad \lim_{x_0} f = \ell.$$



Example 1.17 (Linear Function)

Let $f(x) = 3x - 2$. We want to verify that $\lim_{x\to 1} f(x) = 1$ using the ϵ-δ definition.
Given any $\epsilon > 0$, we need to find $\delta > 0$ such that:

$$0 < |x - 1| < \delta \implies |(3x - 2) - 1| < \epsilon$$

Working backwards:

$$\begin{aligned} |3x - 3| &< \epsilon \\ 3|x - 1| &< \epsilon \\ |x - 1| &< \frac{\epsilon}{3} \end{aligned}$$

Thus, choosing $\delta = \frac{\epsilon}{3}$ satisfies the definition. For example:

- If $\epsilon = 0.3$, then $\delta = 0.1$
- If $\epsilon = 0.03$, then $\delta = 0.01$

This proves that $\lim_{x\to 1}(3x - 2) = 1$.

Example 1.18 (Quadratic Function)

Consider $f(x) = x^2$ at $x_0 = 2$. We claim $\lim_{x\to 2} x^2 = 4$.
For any $\epsilon > 0$, we need $|x^2 - 4| < \epsilon$ when $0 < |x - 2| < \delta$.
Notice that:

$$|x^2 - 4| = |x - 2||x + 2|$$

If we restrict $\delta \leq 1$, then $1 < x < 3$, so $|x + 2| < 5$. Thus:

$$|x^2 - 4| < 5|x - 2|$$

To make this $< \epsilon$, we need $|x - 2| < \epsilon/5$. Therefore, we take:

$$\delta = \min\left(1, \frac{\epsilon}{5}\right)$$

This shows how the choice of δ may depend on both ϵ and the behavior of the function near x_0.

Infinite Limits

Definition 1.17 (Infinite Limits at a Point)

Let f be defined on a punctured neighborhood of x_0.

- f tends to $+\infty$ as $x \to x_0$ if:

$$\forall M > 0,\ \exists \delta > 0,\ \forall x \in I:\ 0 < |x - x_0| < \delta \implies f(x) > M$$

  We write $\lim_{x \to x_0} f(x) = +\infty$.

- f tends to $-\infty$ as $x \to x_0$ if:

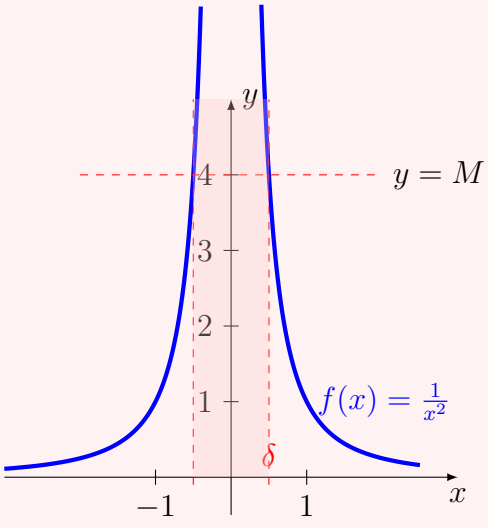
$$\forall M > 0,\ \exists \delta > 0,\ \forall x \in I:\ 0 < |x - x_0| < \delta \implies f(x) < -M$$

  We write $\lim_{x \to x_0} f(x) = -\infty$.

Example 1.19

The function $f(x) = \frac{1}{x^2}$ tends to $+\infty$ as $x \to 0$. For any $M > 0$, choose $\delta = \frac{1}{\sqrt{M}}$. Then:

$$0 < |x| < \delta \implies x^2 < \delta^2 = \frac{1}{M} \implies \frac{1}{x^2} > M$$



1.3.2 Operations on Limits and Limit Theorems

Theorem 1.1 (Limit Arithmetic)

Let f and g be functions with $\lim_{x\to x_0} f(x) = \ell$ and $\lim_{x\to x_0} g(x) = \ell'$ (where x_0 can be finite or $\pm\infty$). Then:

1. **Scalar Multiplication**: $\lim_{x\to x_0}[\lambda f(x)] = \lambda\ell$ for any $\lambda \in \mathbb{R}$
2. **Sum Rule**: $\lim_{x\to x_0}[f(x) + g(x)] = \ell + \ell'$
3. **Product Rule**: $\lim_{x\to x_0}[f(x) \cdot g(x)] = \ell \cdot \ell'$
4. **Quotient Rule**: If $\ell' \neq 0$, $\lim_{x\to x_0}\left[\frac{f(x)}{g(x)}\right] = \frac{\ell}{\ell'}$
5. **Reciprocal Rule**: If $\ell \neq 0$, $\lim_{x\to x_0}\frac{1}{f(x)} = \frac{1}{\ell}$

Example 1.20

Let $f(x) = x^2 + 3x$ and $g(x) = 2x - 1$. Then as $x \to 2$:

$$\lim_{x\to 2} f(x) = 4 + 6 = 10$$
$$\lim_{x\to 2} g(x) = 4 - 1 = 3$$
$$\lim_{x\to 2}\frac{f(x)}{g(x)} = \frac{10}{3} \quad \text{(by Quotient Rule)}$$

Theorem 1.2 (Squeeze Theorem)

If $f(x) \leq g(x) \leq h(x)$ for all x in a punctured neighborhood of x_0, and $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = L$, then $\lim_{x\to x_0} g(x) = L$.

Example 1.21

To find $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$, note that:

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

Since $\lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2 = 0$, by the Squeeze Theorem the limit is 0.

Theorem 1.3 (Continuity and Limits)

If f is continuous at x_0, then $\lim_{x\to x_0} f(x) = f(x_0)$. This applies to all elementary functions (polynomials, rational functions, trigonometric functions, exponential and logarithmic functions, etc.) at points in their domains.

Example 1.22

Since $f(x) = \sin(x)$ is continuous everywhere:

$$\lim_{x\to\pi/2} \sin(x) = \sin\left(\frac{\pi}{2}\right) = 1$$

1.4 Continuity

1.4.1 Continuity at a point

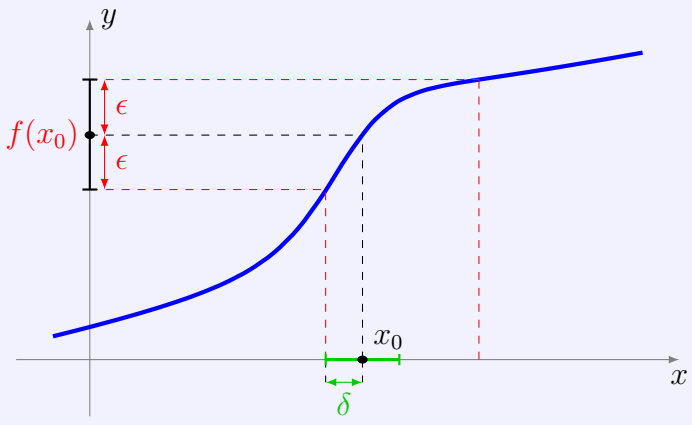
Definition 1.18

Let $f : I \to \mathbb{R}$ be a function defined on the domain I of the real numbers. Let $x_0 \in \mathbb{R}$ be a point in the domain I. We say that the function f is continuous at the point x_0 if the following holds:

$$\forall \epsilon > 0, \quad \exists \delta > 0, \quad \forall x \in I, \quad |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon,$$

we write:

$$\lim_{x\to x_0} f(x) = f(x_0).$$



Example 1.23

The function $f(x) = e^x$ is continuous at the point $x_0 = 0$ because

$$\lim_{x \to x_0} f(x) = \lim_{x \to 0} e^x = e^0 = 1 = f(x_0).$$

1.4.2 Continuity on domain

Definition 1.19

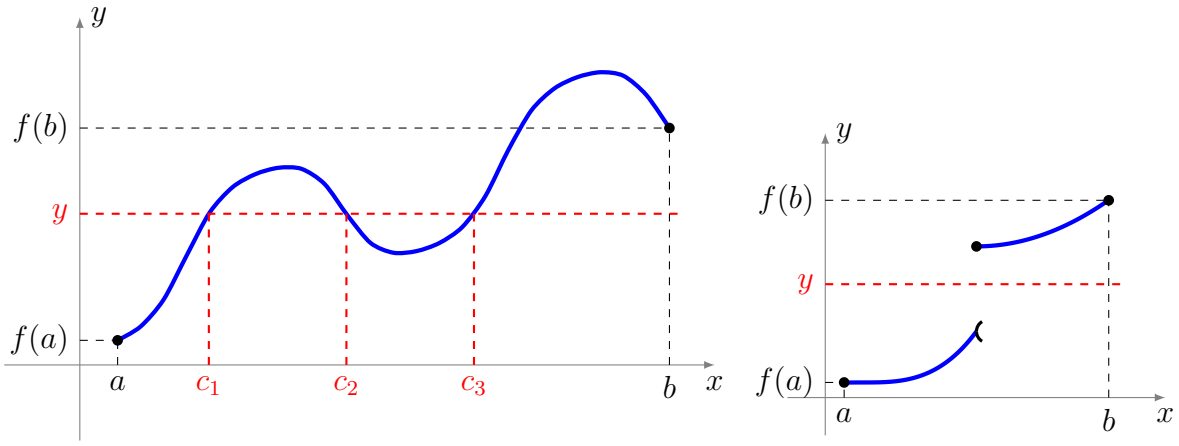
Let $f : I \to \mathbb{R}$ be a function defined on the domain I of $\mathbb{R}$.
We say that the function f is continuous on the domain I if it is continuous on all points of the domain I. We denote the set of continuous functions on the domain of I as $\mathscr{C}(I)$.

Mean Value Theorem

Theorem 1.4

Let $f : [a, b] \to \mathbb{R}$ be a function that is continuous on the closed interval $[a, b]$. For any real number y that lies between $f(a)$ and $f(b)$, there exists a real number $c \in [a, b]$ such that $f(c) = y$.

(In the left figure), the real number c is not necessarily unique. On the other hand, if the function is not continuous, then the theorem does not hold (as shown in the figure on the right).



1.4.3 Continuous extension

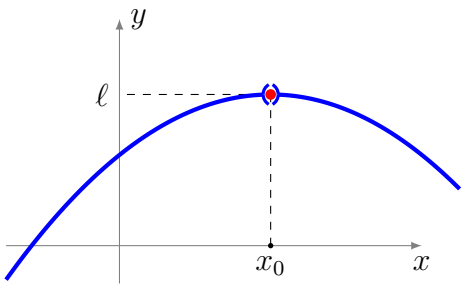
Definition 1.20

Let the domain I, x_0 be the point from I and $f : I_{\{x_0\}} \to \mathbb{R}$ be a function.

1) We say that the function f is continually extendable at the point x_0 if f accepts a finite limit at x_0, and we write:
$$\ell = \lim_{x_0} f.$$

2) We then define the function that we denote $\tilde{f} : I \to \mathbb{R}$ for each $x \in I$
$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq x_0 \\ \ell & \text{if } x = x_0. \end{cases}$$

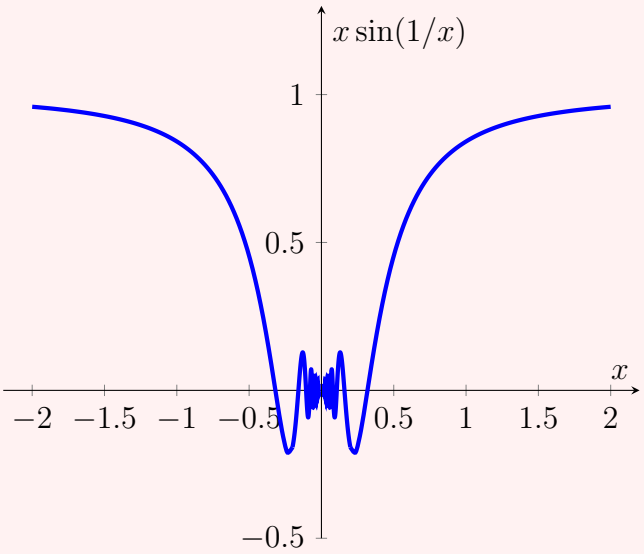
Then the function $\tilde{f}$ is continuous at point x_0, and the extension of the function f is called continuing at point x_0.



Example 1.24

Let the function defined on the set $\mathbb{R}^*$ be as follows
$$f(x) = x \sin\left(\frac{1}{x}\right).$$

Does f accept extension by continuing at 0?
We have for each $x \in \mathbb{R}^*$ that $|f(x)| \leq |x|$, we get that f goes to 0 at 0. That is, it is extendable continuously at 0 and its extension is the function $\tilde{f}$ defined on $\mathbb{R}$ as follows:
$$\tilde{f}(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$



1.4.4 Operations on continuous functions

The primary operations on continuity are immediate consequences of analogous issues at the endpoints.

Proposition 1.4.1. *Let the two functions $f, g : I \to \mathbb{R}$ be given. Let $x_0 \in I$ be a point, hence:*

- *$\lambda \cdot f$ is continuous at x_0 ($\forall \lambda \in \mathbb{R}$).*
- *$f + g$ is continuous at x_0.*
- *$f \cdot g$ is continuous at x_0.*
- *If $f(x_0) \neq 0$, then $\frac{1}{f}$ is continuous at x_0.*

Proposition 1.4.2. *Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be two functions, where $f(I) \subset J$. If f is continuous at the point $x_0 \in I$ and g is continuous at the point $f(x_0)$, then the composite function $g \circ f$ is continuous at the point x_0.*

1.5 Derivative and derivation laws

1.5.1 Derivative at a Point

Definition 1.21 (Differentiability at a Point)

Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ be a function. For $x_0 \in I$, we say f is **differentiable** at x_0 if the following limit exists:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

When this limit exists, it is called the **derivative** of f at x_0 and denoted by $f'(x_0)$ or $\frac{df}{dx}\big|_{x=x_0}$.

Example 1.25 (Linear Function)

For $f(x) = 3x + 2$, the derivative at any point x_0 is:

$$\begin{aligned} f'(x_0) &= \lim_{h \to 0} \frac{[3(x_0 + h) + 2] - [3x_0 + 2]}{h} \\ &= \lim_{h \to 0} \frac{3h}{h} = 3 \end{aligned}$$

This shows the derivative (slope) is constant, as expected for a linear function.

Example 1.26 (Quadratic Function)

For $f(x) = x^2$, the derivative at $x_0 = 1$ is:

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{(1 + h)^2 - 1^2}{h} \\ &= \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h} \\ &= \lim_{h \to 0} (2 + h) = 2 \end{aligned}$$

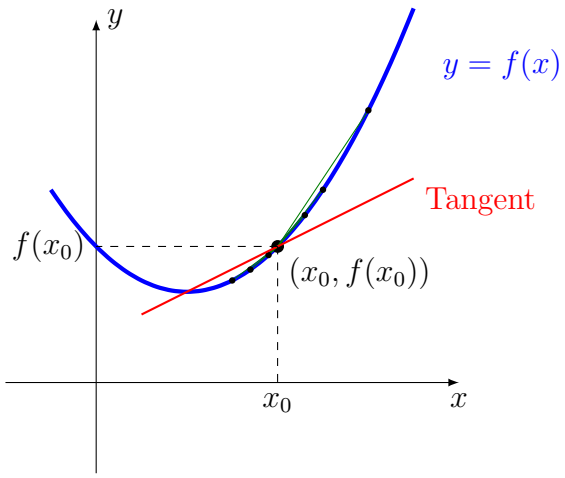
This matches our general formula that $\frac{d}{dx} x^2 = 2x$.

Definition 1.22 (Differentiability on an Interval)

A function f is **differentiable on an open interval** I if it is differentiable at every point $x \in I$. The function $f' : I \to \mathbb{R}$ that maps each x to its derivative $f'(x)$ is called the **derivative function**.

1.5.2 Geometric Interpretation of the Derivative

The derivative has a fundamental geometric interpretation as the slope of the tangent line to the function's graph.



The tangent line at $(x_0, f(x_0))$ has equation:

$$y = f'(x_0)(x - x_0) + f(x_0)$$

Theorem 1.5 (Differentiability Implies Continuity)

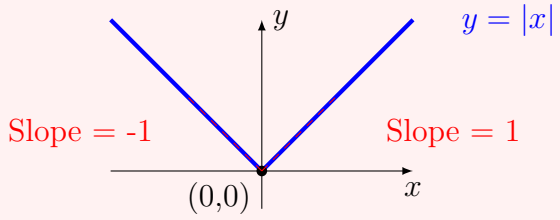
If f is differentiable at x_0, then f is continuous at x_0. The converse is not true - continuity does not imply differentiability.

Example 1.27 (Non-Differentiable Function)

The absolute value function $f(x) = |x|$ is continuous at $x = 0$ but not differentiable there:

$$\lim_{h \to 0^+} \frac{|0+h| - |0|}{h} = 1$$
$$\lim_{h \to 0^-} \frac{|0+h| - |0|}{h} = -1$$

The left and right limits disagree, so the derivative doesn't exist.



Indeed, the rate of increase at $x_0 = 0$ achieves:

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x} = \begin{cases} +1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

1.5.3 Rules of Differentiation

Theorem 1.6 (Basic Differentiation Rules)

Let f, g be differentiable functions and $c \in \mathbb{R}$ a constant:

- **Linearity**:

$$(f + g)' = f' + g'$$
$$(cf)' = cf'$$

- **Product Rule**:

$$(fg)' = f'g + fg'$$

- **Quotient Rule**:

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \quad (g \neq 0)$$

- **Chain Rule**:

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Example 1.28 (Product Rule Application)

Let $f(x) = x^2 \sin(x)$. Then:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^2) \cdot \sin(x) + x^2 \cdot \frac{d}{dx}(\sin(x)) \\ &= 2x \sin(x) + x^2 \cos(x) \end{aligned}$$

Example 1.29 (Chain Rule Application)

For $f(x) = \sqrt{x^2 + 1}$:

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x^2 + 1}} \cdot \frac{d}{dx}(x^2 + 1) \\ &= \frac{2x}{2\sqrt{x^2 + 1}} = \frac{x}{\sqrt{x^2 + 1}} \end{aligned}$$

1.5.4 Derivatives of Elementary Functions

Function	Derivative	Domain of x
c (constant)	0	$\mathbb{R}$
x^n	nx^{n-1}	$\mathbb{R}$ (for $n \in \mathbb{N}$)
e^x	e^x	$\mathbb{R}$
a^x	$a^x \ln a$	$\mathbb{R}$
$\ln x$	$\frac{1}{x}$	$x > 0$
$\log_a x$	$\frac{1}{x \ln a}$	$x > 0$
$\sin x$	$\cos x$	$\mathbb{R}$
$\cos x$	$-\sin x$	$\mathbb{R}$
$\tan x$	$\sec^2 x$	$x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$
$\sinh x$	$\cosh x$	$\mathbb{R}$
$\cosh x$	$\sinh x$	$\mathbb{R}$
$\tanh x$	${}^2x$	$\mathbb{R}$

Example 1.30 (Higher Order Derivatives)

For $f(x) = x^4 + 3x^2 - 2x$:

$$\begin{aligned} f'(x) &= 4x^3 + 6x - 2 \\ f''(x) &= 12x^2 + 6 \\ f'''(x) &= 24x \\ f^{(4)}(x) &= 24 \\ f^{(n)}(x) &= 0 \quad \text{for } n \geq 5 \end{aligned}$$

Theorem 1.7 (Differentiation of Inverse Functions)

If f is bijective and differentiable at x_0 with $f'(x_0) \neq 0$, then its inverse f^{-1} is differentiable at $y_0 = f(x_0)$ and:

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

Example 1.31 (Derivative of Inverse)

For $f(x) = e^x$, its inverse is $f^{-1}(x) = \ln x$. Then:

$$(\ln x)' = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

1.6 Successive Derivatives

Definition 1.23 (Higher-Order Derivatives)

Let $f : I \to \mathbb{R}$ be a differentiable function. We define the successive derivatives recursively:

- $f^{(0)} = f$ (the zeroth derivative is the function itself)
- $f^{(1)} = f'$ (the first derivative)
- $f^{(2)} = f'' = (f')'$ (the second derivative)
- $f^{(n+1)} = \left(f^{(n)}\right)'$ for $n \geq 0$ (the $(n+1)$-th derivative)

If $f^{(n)}$ exists, we say f is **n-times differentiable**. If all orders exist, f is **infinitely differentiable** or **smooth**.

Example 1.32 (Polynomial Function)

For $f(x) = x^5 - 3x^2 + 2$, we have:

$$\begin{aligned} f'(x) &= 5x^4 - 6x \\ f''(x) &= 20x^3 - 6 \\ f'''(x) &= 60x^2 \\ f^{(4)}(x) &= 120x \\ f^{(5)}(x) &= 120 \\ f^{(n)}(x) &= 0 \quad \text{for } n \geq 6 \end{aligned}$$

This shows that polynomials have finite differentiability orders.

Example 1.33 (Exponential Function)

For $f(x) = e^{2x}$, all derivatives exist:

$$\begin{aligned} f'(x) &= 2e^{2x} \\ f''(x) &= 4e^{2x} \\ f'''(x) &= 8e^{2x} \\ &\vdots \\ f^{(n)}(x) &= 2^n e^{2x} \end{aligned}$$

The exponential function is infinitely differentiable.

Leibniz's Rule for Higher Derivatives

The following theorem generalizes the product rule to higher derivatives:

Theorem 1.8 (Leibniz's Rule)

Let $f, g : I \to \mathbb{R}$ be n-times differentiable functions. Then:

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

where $\binom{n}{k}$ is the binomial coefficient and $f^{(0)} = f$.

Example 1.34 (Application of Leibniz's Rule)

Compute the 4th derivative of $f(x) = x^2 e^x$.
Let $g(x) = x^2$ and $h(x) = e^x$. Then:

$$\begin{aligned}(g \cdot h)^{(4)} &= \sum_{k=0}^{4} \binom{4}{k} g^{(k)} h^{(4-k)} \\ &= \binom{4}{0} x^2 e^x + \binom{4}{1} (2x) e^x + \binom{4}{2} (2) e^x \\ &= x^2 e^x + 8x e^x + 12 e^x \\ &= e^x (x^2 + 8x + 12)\end{aligned}$$

Note that $g^{(k)} = 0$ for $k \geq 3$.

Important Properties

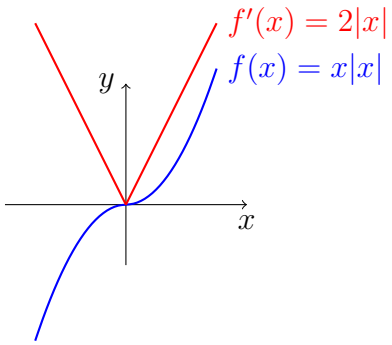
Proposition 1.6.1 (Linearity of Higher Derivatives)**.** *For n-times differentiable functions f, g and constants a, b:*

$$(af + bg)^{(n)} = af^{(n)} + bg^{(n)}$$

Proposition 1.6.2 (Derivative of Composition)**.** *For f and g sufficiently differentiable, the chain rule extends to higher derivatives, though the general formula (Faà di Bruno's formula) is more complex than Leibniz's rule.*

Example 1.35 (Non-Example: Differentiability)

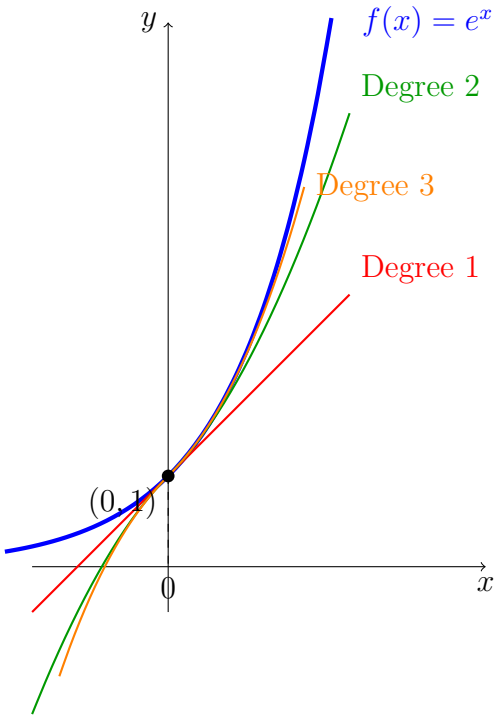
The function $f(x) = x|x|$ is differentiable everywhere, but its derivative $f'(x) = 2|x|$ is not differentiable at $x = 0$. Thus, f is once but not twice differentiable at 0.



1.7 Limited Expansions and Taylor Approximations

1.7.1 Introduction to Limited Expansions

The basic idea is to approximate a function $f(x)$ near a point x_0 using a polynomial that matches the function's value and derivatives at x_0.



1.7.2 Taylor's Theorem

Theorem 1.9 (Taylor's Theorem with Peano Remainder)

Let $f : I \to \mathbb{R}$ be n-times differentiable at $x_0 \in I$. Then there exists a function ε such that:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + (x - x_0)^n \varepsilon(x - x_0)$$

where $\lim_{x \to x_0} \varepsilon(x - x_0) = 0$.

Definition 1.24 (Taylor Polynomial)

The **Taylor polynomial** of degree n for f at x_0 is:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

The remainder term is $R_n(x) = f(x) - P_n(x)$.

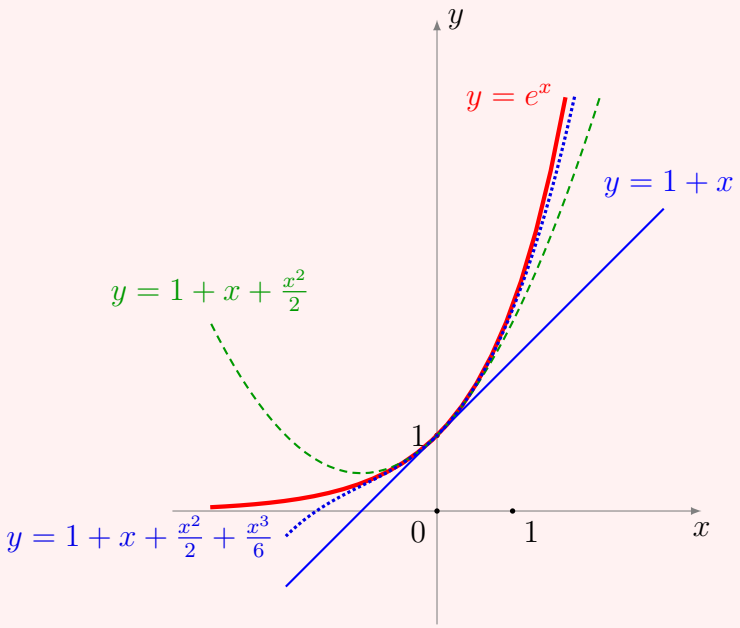
Example 1.36 (Exponential Function)

For $f(x) = e^x$ at $x_0 = 0$:

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= 1 \\ f''(0) &= 1 \\ &\vdots \\ f^{(n)}(0) &= 1 \end{aligned}$$

Thus the Taylor polynomial is:

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$



1.7.3 Maclaurin Series (Special Case at $x_0 = 0$)

When the expansion point is $x_0 = 0$, the Taylor series is called a Maclaurin series.

Theorem 1.10 (Maclaurin Series)

For f infinitely differentiable near 0:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

when the series converges.

Common Maclaurin Expansions

Function	Maclaurin Expansion	Domain of x
e^x	$\sum_{k=0}^{\infty} \frac{x^k}{k!}$	$\mathbb{R}$
$\sin x$	$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$	$\mathbb{R}$
$\cos x$	$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$	$\mathbb{R}$
$\ln(1+x)$	$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$	$-1 < x \leq 1$
$(1+x)^\alpha$	$\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$	$\lvert x\rvert < 1$ (convergence)
$\frac{1}{1-x}$	$\sum_{k=0}^{\infty} x^k$	$\lvert x\rvert < 1$

Example 1.37 (Sine and Cosine Functions)

For $f(x) = \sin x$:

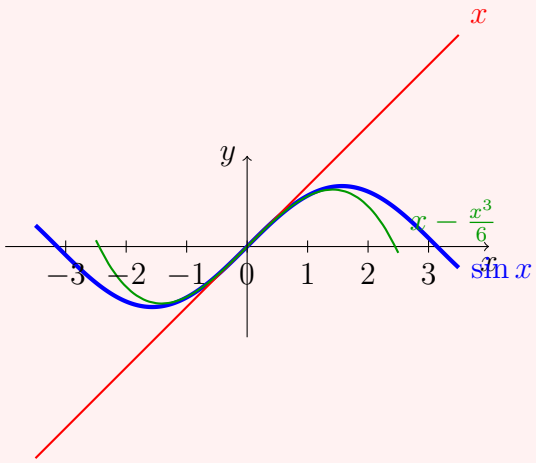
$$\begin{aligned} f(0) &= 0 \\ f'(0) &= 1 \\ f''(0) &= 0 \\ f'''(0) &= -1 \\ f^{(4)}(0) &= 0 \\ &\vdots \end{aligned}$$

Thus:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Similarly for $\cos x$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$



1.7.4 Operations on Limited Expansions

When working with limited expansions, we can perform various operations while maintaining the approximation order.

Proposition 1.7.1 (Algebraic Operations)**.** *Let $f(x) = P_n(x) + o(x^n)$ and $g(x) = Q_n(x) + o(x^n)$ be limited expansions at 0.*

- ***Sum**: $(f+g)(x) = P_n(x) + Q_n(x) + o(x^n)$*
- ***Product**: $(f \cdot g)(x) = P_n(x)Q_n(x) + o(x^n)$ (truncate to degree n)*
- ***Composition**: If $g(0) = 0$, then $(f \circ g)(x) = P_n(Q_n(x)) + o(x^n)$*
- ***Division**: For $g(0) \neq 0$, $\frac{1}{g(x)} = \frac{1}{Q_n(x)} + o(x^n)$*

Example 1.38 (Product of Expansions)

Compute the limited expansion of $e^x \cos x$ at order 3:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3) \\ \cos x &= 1 - \frac{x^2}{2} + o(x^3) \\ e^x \cos x &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\left(1 - \frac{x^2}{2}\right) + o(x^3) \\ &= 1 + x + \left(\frac{1}{2} - \frac{1}{2}\right)x^2 + \left(\frac{1}{6} - \frac{1}{2}\right)x^3 + o(x^3) \\ &= 1 + x - \frac{x^3}{3} + o(x^3) \end{aligned}$$

1.7.5 Applications and Examples

Example 1.39 (Arctangent Function)

Compute the limited expansion of $\arctan x$ at order 5:

$$\begin{aligned}\frac{d}{dx}\arctan x &= \frac{1}{1+x^2} = 1 - x^2 + x^4 + o(x^5)\\ \arctan x &= \int (1 - x^2 + x^4)dx + o(x^6)\\ &= x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^6)\end{aligned}$$

Example 1.40 (Tangent Function)

Compute $\tan x$ at order 5 using $\tan x = \frac{\sin x}{\cos x}$:

$$\begin{aligned}\sin x &= x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^6)\\ \cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)\\ \frac{1}{\cos x} &= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + o(x^5)\\ \tan x &= \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\left(1 + \frac{x^2}{2} + \frac{5x^4}{24}\right) + o(x^5)\\ &= x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^5)\end{aligned}$$

Example 1.41 (Composite Function)

Compute $\sin(\ln(1+x))$ at order 3:

$$\begin{aligned}\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)\\ \sin u &= u - \frac{u^3}{6} + o(u^3)\\ \sin(\ln(1+x)) &= \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) - \frac{x^3}{6} + o(x^3)\\ &= x - \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)\end{aligned}$$

1.8 Exercises: Real Functions and Limited Expansions