Study of Variation and Graphical Representation: $f(x) = \frac{1}{x^3}$

1 Complex Study of the Function $f(x) = \frac{1}{x^3}$

1.1 Domain of Definition and Parity

The function $f(x) = \frac{1}{x^3} = x^{-3}$ is defined for all real numbers $x \neq 0$:

$$\mathcal{D}_f = \mathbb{R}^* =]-\infty, 0[\cup]0, +\infty[$$

For all $x \in \mathbb{R}^*$:

$$f(-x) = \frac{1}{(-x)^3} = \frac{1}{-x^3} = -\frac{1}{x^3} = -f(x)$$

The function is **odd**. Its representative curve $\mathcal{C}_f$ has the origin $O(0,0)$ as a **center of symmetry**.

1.2 First Derivative and Monotonicity

The derivative of f is given by:

$$f'(x) = \frac{d}{dx}\left(x^{-3}\right) = -3x^{-4} = -\frac{3}{x^4}$$

Since $x^4 > 0$ for all $x \in \mathbb{R}^*$, we strictly have:

$$f'(x) < 0, \quad \forall x \in \mathcal{D}_f$$

Therefore, the function f is **strictly decreasing** on $]-\infty, 0[$ and **strictly decreasing** on $]0, +\infty[$.

1.3 Limits at Boundaries and Asymptotes

- **At $-\infty$ and $+\infty$:**

  $$\lim_{x \to -\infty} \frac{1}{x^3} = 0^- \quad \text{and} \quad \lim_{x \to +\infty} \frac{1}{x^3} = 0^+$$

  The line with equation $y = 0$ (the x-axis) is a **horizontal asymptote**.

- **At 0^- and 0^+:**

  $$\lim_{x \to 0^-} \frac{1}{x^3} = -\infty \quad \text{and} \quad \lim_{x \to 0^+} \frac{1}{x^3} = +\infty$$

  The line with equation $x = 0$ (the y-axis) is a **vertical asymptote**.

1.4 Convexity (Second Derivative)

$$f''(x) = \frac{d}{dx}\left(-\frac{3}{x^4}\right) = \frac{12}{x^5}$$

- For $x < 0$, $f''(x) < 0$: the function is **concave** on $]-\infty, 0[$.
- For $x > 0$, $f''(x) > 0$: the function is **convex** on $]0, +\infty[$.

2 Variation Table

x	$-\infty$		0		$+\infty$
$f'(x)$		$-$	$\|\|$	$-$	
$f''(x)$		$-$	$\|\|$	$+$	
$f(x)$	0	$\searrow$	$\|\|$ $-\infty$	$\searrow$	0

3 Graphical Representation

