Mathematics Department, University of Biskra
Module: Analysis 2

First year license
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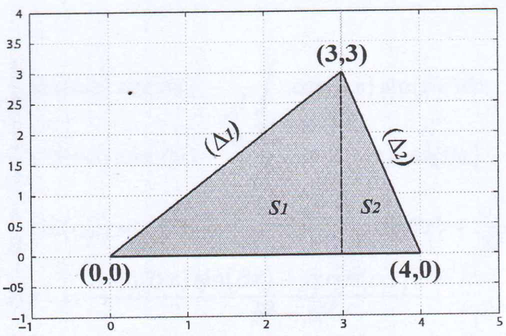
Solution of the Final Exam

Solution of the Exercise 1

i) Using the decomposition technique of the rational function we get:

$$\begin{aligned}
\int \frac{2x-1}{x^2-4x+13} &= \int \frac{(2x-4)+3}{x^2-4x+13}dx \\
&= \int \frac{(x^2-4x+13)'}{x^2-4x+13}dx + \int \frac{3}{(x-2)^2+3^2}dx \\
&= \ln(x^2-4x+13) + \arctan\left(\frac{x-2}{3}\right) + c
\end{aligned}$$

ii) We note that $S = s_1 + s_2$ (see the figure below) where s_1 and s_2 are delimited by the lines (Δ_1) and (Δ_2) respectively.



- The equation of the line passed on the points $(0,0)$ and $(3,3)$ is given by
$$(\Delta_1) : y = x.$$
- The equation of the line passed on the points $(4,0)$ and $(3,3)$ is given by
$$(\Delta_1) : y = -3x + 12.$$

Consequently,

$$\begin{aligned}
S &= \int_0^3 |x|dx + \int_3^4 |-3x+12|dx = \int_0^3 xdx + \int_3^4 -3x+12dx \\
&= \frac{1}{2}x^2\Big|_0^3 + \frac{-3}{2}x^2 + 12x\Big|_3^4 = \frac{9}{2} + \frac{3}{2} \\
&= 6.
\end{aligned}$$

iii) Let us note

$$I(x) = \int \sin(\alpha x)\, \cos(\beta x)\, dx, \text{ with } \alpha,\ \beta \in \mathbb{R}^*.$$

For the current integral, we distinguish two cases, namely: the case where $\alpha = \beta$ and the case where $\alpha \neq \beta$.

Case $\alpha = \beta$**:** In this case, we have

$$I(x) = \int \sin(\alpha x)\ \cos(\beta x)\, dx = \int \sin(\alpha x)\ \cos(\alpha x)\, dx.$$

If we take the following:

$$\begin{cases} u & = \sin(\alpha x) \\ v' & = \cos(\alpha x) \end{cases} \Longrightarrow \begin{cases} u' & = \alpha\cos(\alpha x), \\ v & = \frac{1}{\alpha}\sin(\alpha x), \end{cases}$$ 0,1 ✓

then we obtain:

$$I(x) = \frac{1}{\alpha}\sin^2(\alpha x) - I(x) \Longrightarrow \frac{1}{2\alpha}\sin^2(\alpha x) + c, \text{ with } c \in \mathbb{R}.$$ 0,1 ✓

Case $\alpha \neq \beta$**:** In this case, we must use integration by parts twice in succession to obtain the expression for the integral in question. If we choose, for the first integration by parts

$$\begin{cases} u & = \sin(\alpha x) \\ v' & = \cos(\beta x) \end{cases} \Longrightarrow \begin{cases} u' & = \alpha\cos(\alpha x) \\ v & = \frac{1}{\beta}\sin(\beta x) \end{cases},$$ 0,1 ✓

and for the second integration by parts

$$\begin{cases} u & = \cos(\alpha x) \\ v' & = \sin(\beta x) \end{cases} \Longrightarrow \begin{cases} u' & = -\alpha\sin(\alpha x) \\ v & = -\frac{1}{\beta}\cos(\beta x) \end{cases}$$ 0,1 ✓

then,

$$\begin{aligned} I(x) &= \frac{1}{\beta}\sin(\alpha x)\sin(\beta x) - \frac{\alpha}{\beta}\int \cos(\alpha x)\sin(\beta x)dx \\ &= \frac{1}{\beta}\sin(\alpha x)\sin(\beta x) - \frac{\alpha}{\beta}\left[\frac{-1}{\beta}\cos(\alpha x)\cos(\beta x) - \frac{\alpha}{\beta}I(x)\right] \\ &= \frac{1}{\beta}\sin(\alpha x)\sin(\beta x) + \frac{\alpha}{\beta^2}\cos(\alpha x)\cos(\beta x) + \frac{\alpha^2}{\beta^2}I(x) \\ \Longrightarrow\ I(x) &= \frac{\beta\sin(\alpha x)\sin(\beta x) + \alpha\cos(\alpha x)\cos(\beta x)}{\beta^2 - \alpha^2} + c \text{ with } c \in \mathbb{R}. \end{aligned}$$ 0,1 ✓ 0,1 ✓

Solution of the Exercise 2 Consider the following first-order ordinary differential equation.

$$y' - \left(\frac{2+x}{x}\right)y = x^2\sqrt{y}. \tag{4}$$

1. We have

$$u = y/x^2 \Longrightarrow y = x^2 u \text{ and } y' = x^2 u' + 2xu.$$ 0,1 ✓

Thus, by substituting y and y' into equation (4), we obtain:

$$(x^2u' + 2xu) - \left(\frac{2+x}{x}\right)(x^2 u) = x^2\sqrt{x^2 u} \Longrightarrow u' - u = x\sqrt{u}.$$ 0,1 ✓

2. Now, let's consider the following equation

$$u' - u = x\sqrt{u}. \tag{5}$$

We note that equation (5) is a Bernoulli equation with $n = \frac{1}{2}$.

To solve this problem, we take $z = u^{\frac{1}{2}}$, so $2z' = u'u^{\frac{1}{2}}$, and the equation becomes a linear differential equation. 0,1 ✓

$$u'u^{\frac{1}{2}} - u^{\frac{1}{2}} = x \Longrightarrow 2z' - z = x. \tag{6}$$ 0,1 ✓

To solve equation (6), we first determine its homogeneous solution z_h. Then, using the constant-variation technique, we can determine its global solution z_g.

z_h

$$\begin{aligned} 2z' - z = 0 \quad &\Longrightarrow \quad \frac{z'}{z} = \frac{1}{2} \\ &\Longrightarrow \quad \int \frac{1}{z} dz = \int \frac{1}{2} dx \\ &\Longrightarrow \quad z = K e^{\frac{x}{2}}, \text{ with } K \in \mathbb{R}_+. \end{aligned}$$

$z_g = ?$ To find the global solution we use the variation of the constant technique.

$$z = K(x) e^{\frac{x}{2}} \Longrightarrow z' = K'(x) e^{\frac{x}{2}} + \frac{1}{2} K(x) e^{\frac{x}{2}}.$$

thus,

$$2K(x)' e^{\frac{x}{2}} + K(x) e^{\frac{x}{2}} - K(x) e^{\frac{x}{2}} = x \Longrightarrow K(x) = \int x e^{-\frac{x}{2}}.$$

To compute $K(x)$, we use integration by parts. For this, we choose the following:

$$\left\{ \begin{array}{lcl} u & = & x \\ v' & = & e^{-\frac{x}{2}} \end{array} \right. \Longrightarrow \left\{ \begin{array}{lcl} u' & = & 1 \\ v & = & -\frac{1}{2} e^{-\frac{x}{2}} \end{array} \right.$$

then

$$K(x) = -x e^{-\frac{x}{2}} + \frac{1}{2} \int e^{-\frac{x}{2}}\, dx = -\frac{1}{2}\left(x + \frac{1}{2}\right) e^{-\frac{x}{2}} + c.$$

Thus, the global solution of the equation (6) is given by:

$$z = \left(-\frac{1}{2}\left(x + \frac{1}{2}\right) e^{-\frac{x}{2}} + c\right) e^{\frac{x}{2}} = c e^{\frac{x}{2}} - \frac{1}{2}\left(x + \frac{1}{2}\right).$$

Consequently, as $z = u^{\frac{1}{2}}$ the global solution of the original equation (5) is given by:

$$u_g = z_g^2 = \left(c e^{\frac{x}{2}} - \frac{1}{2}\left(x + \frac{1}{2}\right)\right)^2.$$

3. From the first and second question we deduce that

$$u = x^2 z = \left(c x e^{\frac{x}{2}} - \frac{x}{2}\left(x + \frac{1}{2}\right)\right)^2.$$

Solution of the Exercise 3 Consider the following second-order ordinary differential equation.

$$y'' + y' - 2y = I(x) + J(x) + K(x), \text{ with } \left\{ \begin{array}{lcl} I(x) & = & 4x, \\ J(x) & = & -3e^{-2x}, \\ K(x) & = & -\frac{2x^2 \ln(x) - x + 1}{x^2}. \end{array} \right. \tag{7}$$

1. We have $y = \ln(x)$ then $y' = \frac{1}{x}$ and $y'' = \frac{-1}{x^2}$. Substituting y, y' and y'' with their expressions into the left-hand side of equation (7), we obtain

$$y'' + y' - 2y = \frac{-1}{x^2} + \frac{1}{x} - 2\ln(x) = K(x),$$

this means that $y = \ln(x)$ is a particular solution of the equation under consideration.

2. Find the global solution y_g of (7).

$y_h =?$ Note that the characteristic equation associated with the given differential equation is :

$$R^2 + R - 2 = 0.$$

Furthermore, it admits two distinct solutions, indeed,

$$\Delta = 9 > 0 \Longrightarrow \begin{cases} R_1 &= \frac{-1-\sqrt{9}}{2} = -2; \\ R_2 &= \frac{-1+\sqrt{9}}{2} = 1; \end{cases}$$

so, the homogenous solution of the equation is expressed as follows:

$$y_h = C_1 e^x + C_2 e^{-2x}.$$

$y_{p_1} =?$ Form the expression of $I(x)$ we conclude that the form of the particular solution associated with is $y_{p_1} = ax + b$.
as $y_{p_1} = ax + b$ then $y' = a$ and $y'' = 0$. So,

$$0 + a - 2ax + 2b = 4x \Longrightarrow -2ax + (a + 2b) = 4x + 0 \Longrightarrow \begin{cases} a &= -2 \\ b &= 1 \end{cases}$$

Consequently, the particular solution of the equation associated with $4x$ is

$$y_{p_1} = -2ax + 1.$$

$y_{p_2} =?$ The general form of $J(x) = -3e^{-2x}$ is ae^{-2x}, but as this last is already a solution (see the homogeneous solution) so the particular solution that associated with $J(x)$ will be $y_{p_2} = axe^{-2x}$.

$$y_{p_2} = axe^{-2x} \Longrightarrow y' = (a - 2ax)e^{-2x} \text{ and } y'' = (-4a + 4ax)e^{-2x}.$$

hence,

$$\left((-4a + 4ax)e^{-2x}\right) + \left((a - 2ax)e^{-2x}\right) - 2\left(axe^{-2x}\right) = -3e^{-2x} \Longrightarrow -3a = -3 \Longrightarrow a = 1.$$

This means that the particular solution of the equation associated with $-3e^{-2ax}$ is

$$y_{p_2} = xe^{-2ax}.$$

$y_{p_3} =?$ According to the first question, we have already shown that the particular solution associated with $K(x)$ is $y = \ln(x)$; so this last one therefore represents the third particular solution i.e.

$$y_{p_3} = \ln(x).$$

According to the previous results, we conclude that the global solution of the differential equation under consideration is expressed as follows:

$$y_g = y_h + y_{p_1} + y_{p_2} + y_{p_3} = y_h = C_1 e^x + C_2 e^{-2x} - 2x - 1 + xe^{-2x} + \ln(x).$$