Correction of Tutorial Series N°05: Matrix Reduction

Exercise 1

Given the matrix $A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$.

1) Characteristic polynomial $P_A(\lambda)$:

$$\begin{aligned} P_A(\lambda) = \det(A - \lambda I) &= \det \begin{pmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix} \\ &= (4-\lambda)(3-\lambda) - (2)(1) \\ &= 12 - 4\lambda - 3\lambda + \lambda^2 - 2 \\ &= \lambda^2 - 7\lambda + 10 \end{aligned}$$

2) Eigenvalues of A:
The eigenvalues are the roots of the characteristic equation $P_A(\lambda) = 0$:

$$\lambda^2 - 7\lambda + 10 = 0$$

Using the discriminant $\Delta = (-7)^2 - 4(1)(10) = 49 - 40 = 9$, we get:

$$\lambda_1 = \frac{7+3}{2} = 5, \quad \lambda_2 = \frac{7-3}{2} = 2$$

Thus, the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = 2$.

3) Eigenvectors:
Let $V = \begin{pmatrix} x \\ y \end{pmatrix}$ be an eigenvector. We solve $(A - \lambda I)V = 0$.

For $\lambda_1 = 5$:

$$\begin{pmatrix} 4-5 & 2 \\ 1 & 3-5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives the equation $-x + 2y = 0 \implies x = 2y$.
Setting $y = 1$, we get $x = 2$. The first eigenvector is $V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 2$:

$$\begin{pmatrix} 4-2 & 2 \\ 1 & 3-2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives the equation $x + y = 0 \implies x = -y$.
Setting $y = -1$, we get $x = 1$. The second eigenvector is $V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Exercise 2

Given the matrix $B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 1 & 1 \end{pmatrix}$.

1) Eigenvalues without calculation:
Notice that matrix B is a **lower triangular matrix** (all entries above the main diagonal are zero). According to the properties of triangular matrices, the determinant $\det(B - \lambda I)$ is simply

the product of the diagonal elements. Therefore, the eigenvalues are directly the values on the main diagonal:

$$\lambda_1 = 2, \quad \lambda_2 = 3, \quad \lambda_3 = 1$$

2) Diagonalizability of B:

Yes, all three eigenvalues are distinct real numbers ($1 \neq 2 \neq 3$).

Conclusion: Since B is a 3×3 matrix and has exactly 3 distinct eigenvalues, the algebraic multiplicity of each eigenvalue equals its geometric multiplicity ($1 = 1$). Thus, by theorem, matrix B is **diagonalizable**.

3) Eigenvector for the smallest eigenvalue:

The smallest eigenvalue is $\lambda_3 = 1$. We solve $(B - 1I)V = 0$ where $V = (x, y, z)^T$.

$$\begin{pmatrix} 2-1 & 0 & 0 \\ 1 & 3-1 & 0 \\ -1 & 1 & 1-1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives the system of equations:

$$\begin{aligned} 1x = 0 &\implies x = 0 \\ x + 2y = 0 &\implies 0 + 2y = 0 \implies y = 0 \\ -x + y + 0z = 0 &\implies 0 + 0 = 0 \end{aligned}$$

Notice that z is a free variable (it can be any real number). Setting $z = 1$, we get the eigenvector:

$$V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Exercise 3

Given the matrix $C = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

1) Characteristic polynomial $P_C(\lambda)$:

$$\begin{aligned} P_C(\lambda) = \det(C - \lambda I) = \det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} \\ = (1-\lambda)^2 - (2)(2) \\ = 1 - 2\lambda + \lambda^2 - 4 \\ = \lambda^2 - 2\lambda - 3 \end{aligned}$$

2) Cayley-Hamilton theorem and equation:
The Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation, meaning $P_C(C) = 0$ (the zero matrix).
Substituting C into the polynomial, we get:

$$C^2 - 2C - 3I = 0$$

where I is the 2×2 identity matrix and 0 is the 2×2 zero matrix.

3) Deduce the inverse matrix C^{-1}:
We start from the Cayley-Hamilton equation:

$$C^2 - 2C = 3I$$

Factor out C on the left side:

$$C(C - 2I) = 3I$$

Divide both sides by 3 to isolate I:

$$C \left[\frac{1}{3}(C - 2I) \right] = I$$

By the definition of the inverse matrix ($C \cdot C^{-1} = I$), the term inside the brackets is exactly C^{-1}:

$$C^{-1} = \frac{1}{3}(C - 2I)$$

Now we just perform the basic matrix arithmetic:

$$\begin{aligned} C^{-1} &= \frac{1}{3} \left[\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right] \\ C^{-1} &= \frac{1}{3} \begin{pmatrix} 1-2 & 2-0 \\ 2-0 & 1-2 \end{pmatrix} \\ C^{-1} &= \frac{1}{3} \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix} \end{aligned}$$

This provides the exact inverse without needing to calculate the determinant and adjugate explicitly!