Chapter 5: Matrix Reduction
A Deep Dive into Eigen Theory and Applications

Advanced Linear Algebra

Contents

1 Eigenvalues and Eigenvectors

The foundation of matrix reduction lies in finding invariant directions under a linear transformation. These directions are defined by eigenvectors, and the scaling factors are the eigenvalues.

Definition 1.1 (Eigenvalues and Eigenvectors)**.** Let $A \in \mathcal{M}_n(\mathbb{K})$. A scalar $\lambda \in \mathbb{K}$ is an **eigenvalue** of A if there exists a non-zero vector $v \in \mathbb{K}^n$ such that $Av = \lambda v$. The vector v is the **eigenvector**. The set $E_\lambda = \{v \in \mathbb{K}^n \mid Av = \lambda v\}$ is the **eigenspace** associated with λ.

Example 1.1 (Computing Eigen-pairs for a 2×2 Matrix)**.** Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

Detailed Solution. **1. Eigenvalues:** Solve $\det(A - \lambda I) = 0$.
$(3-\lambda)^2 - 1 = 0 \implies \lambda^2 - 6\lambda + 8 = 0 \implies (\lambda - 4)(\lambda - 2) = 0$.
The eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 2$.

2. Eigenvectors:

For $\lambda_1 = 4$: $(A - 4I)v = 0 \implies \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x = y$. Eigenvector $v_1 = (1, 1)^T$.

For $\lambda_2 = 2$: $(A - 2I)v = 0 \implies \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x = -y$. Eigenvector $v_2 = (1, -1)^T$.

Example 1.2 (Eigenvalues of a Triangular Matrix)**.** Find the eigenvalues of $B = \begin{pmatrix} 5 & -2 & 4 \\ 0 & 3 & 7 \\ 0 & 0 & -1 \end{pmatrix}$.

Detailed Solution. For any upper or lower triangular matrix, the determinant $\det(B - \lambda I)$ is simply the product of its diagonal entries: $P_B(\lambda) = (5-\lambda)(3-\lambda)(-1-\lambda) = 0$. Thus, the eigenvalues can be read directly from the main diagonal: $\lambda_1 = 5, \lambda_2 = 3, \lambda_3 = -1$.

2 Characteristic Polynomial and Cayley-Hamilton Theorem

Theorem 2.1 (Cayley-Hamilton Theorem)**.** *Every square matrix A is a root of its own characteristic polynomial $P_A(X)$. If $P_A(X) = \det(A - XI_n)$, then $P_A(A) = 0_n$.*

Example 2.1 (Using Cayley-Hamilton to find A^{-1})**.** Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$. Find A^{-1} using the theorem.

Detailed Solution. First, find $P_A(X) = \det \begin{pmatrix} 2-X & 1 \\ -1 & 3-X \end{pmatrix} = (2-X)(3-X)+1 = X^2 - 5X + 7$.
By Cayley-Hamilton, $A^2 - 5A + 7I = 0$.
Rewrite to isolate I: $7I = 5A - A^2 = A(5I - A)$.
Multiply by A^{-1} on both sides: $7A^{-1} = 5I - A \implies A^{-1} = \frac{1}{7}(5I - A)$.
$A^{-1} = \frac{1}{7}\left[\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\right] = \frac{1}{7}\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$.

Example 2.2 (Using Cayley-Hamilton to compute high powers)**.** Let A have $P_A(X) = X^2 - 3X + 2$. Express A^3 as a linear combination of A and I.

Detailed Solution. By Cayley-Hamilton: $A^2 - 3A + 2I = 0 \implies A^2 = 3A - 2I$.
To find A^3, multiply by A: $A^3 = 3A^2 - 2A$.
Substitute A^2 back into the equation: $A^3 = 3(3A-2I)-2A = 9A-6I-2A = 7A-6I$.
This allows us to compute any power of A without matrix multiplication!

3 Characterization of Diagonalizable Matrices

Theorem 3.1. *A matrix $A \in \mathcal{M}_n(\mathbb{K})$ is diagonalizable if and only if: 1. $P_A(X)$ is split over $\mathbb{K}$. 2. For every eigenvalue λ_i, the algebraic multiplicity equals the geometric multiplicity: $m(\lambda_i) = \dim(E_{\lambda_i})$.*

Example 3.1 (A Diagonalizable Matrix)**.** Show that $A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$ is diagonalizable and find D.

Detailed Solution. $P_A(\lambda) = (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$.
The eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -1$.
Since we have 2 distinct real eigenvalues for a 2×2 matrix, the algebraic multiplicity of each is 1, which inherently matches the geometric multiplicity. Thus, A is diagonalizable.
The diagonal matrix is $D = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$.

Example 3.2 (A Non-Diagonalizable Matrix)**.** Show that $B = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ is NOT diagonalizable.

Detailed Solution. $P_B(\lambda) = (2-\lambda)^2$. The only eigenvalue is $\lambda = 2$ with an algebraic multiplicity of $m(2) = 2$.
Let's find the geometric multiplicity (dimension of eigenspace E_2).
Solve $(B - 2I)v = 0 \implies \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies y = 0$.
The eigenvectors are of the form $(x, 0)^T = x(1, 0)^T$.

The dimension of E_2 is 1. Since $1 \neq 2$ (Geometric Mult. < Algebraic Mult.), matrix B cannot be diagonalized.

4 Characterization of Trigonalizable Matrices

When a matrix cannot be diagonalized, we aim for the next best thing: Trigonalization (Schur's Form), where $A = PTP^{-1}$ and T is upper triangular.

Theorem 4.1. *A matrix A is trigonalizable over $\mathbb{K}$ if and only if its characteristic polynomial $P_A(X)$ splits completely into linear factors over $\mathbb{K}$.*

Example 4.1 (Trigonalizing a Defective Matrix)**.** Find a triangular form for $A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$.

Detailed Solution. $P_A(\lambda) = (1-\lambda)(3-\lambda) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$.
Eigenvalue $\lambda = 2$. Eigenvector $v_1 = (-1, 1)^T$. Since there is only one independent eigenvector, A is not diagonalizable.
To trigonalize, we complete the basis with a vector $v_2 = (1, 0)^T$.
Transition matrix $P = (v_1, v_2) = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$. $P^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.
Calculate $T = P^{-1}AP = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Matrix T is upper triangular!

Example 4.2 (Trigonalization over $\mathbb{C}$ vs $\mathbb{R}$)**.** Consider $C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (A 90-degree rotation matrix).

Detailed Solution. $P_C(\lambda) = \lambda^2 + 1$.
Over the real numbers $\mathbb{R}$, this polynomial does not split. Hence, C is neither diagonalizable nor trigonalizable in $\mathbb{R}$.
However, over the complex numbers $\mathbb{C}$, $P_C(\lambda) = (\lambda - i)(\lambda + i)$.
Because it splits completely over $\mathbb{C}$, it is trigonalizable (and in fact, diagonalizable because the roots are distinct) over the complex field. This explains oscillatory behavior in continuous systems.

5 Applications of Reduction

Example 5.1 (Application 1: Fast Computation of A^n)**.** Compute A^n for $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$.

Detailed Solution. From diagonalization, we find $P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, and $P^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$.

Using the property $A^n = PD^nP^{-1}$:

$$A^n = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3^n & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$A^n = \begin{pmatrix} 2 \cdot 3^n & 2^n \\ 3^n & 2^n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 3^n - 2^n & -2 \cdot 3^n + 2^{n+1} \\ 3^n - 2^n & -3^n + 2^{n+1} \end{pmatrix}$$

Example 5.2 (Application 2: Solving Coupled Differential Equations)**.** Solve the dynamical system: $\begin{cases} x'(t) = 5x(t) - y(t) \\ y'(t) = -x(t) + 5y(t) \end{cases}$

Detailed Solution. The system matrix is $A = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$.
Eigenvalues: $\det(A - \lambda I) = (5 - \lambda)^2 - 1 = 0 \implies \lambda_1 = 6, \lambda_2 = 4$.
Eigenvectors: $v_1 = (-1, 1)^T$, $v_2 = (1, 1)^T$.
The general solution to $\mathbf{X}' = A\mathbf{X}$ is $\mathbf{X}(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$.

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{6t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus, $x(t) = -c_1 e^{6t} + c_2 e^{4t}$ and $y(t) = c_1 e^{6t} + c_2 e^{4t}$.