Solutions: Tutorial Series N°04

Exercise 1 Solution (Cramer Method)

The system is: $\begin{cases} x + 2y + 3z = 1 \\ -2x - 4y - 5z = 2 \\ 3x + 5y + 6z = 3 \end{cases}$

1) **Matrix Form:**

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -4 & -5 \\ 3 & 5 & 6 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

2) **Determinant of A:**

$$\det(A) = 1\begin{vmatrix} -4 & -5 \\ 5 & 6 \end{vmatrix} - 2\begin{vmatrix} -2 & -5 \\ 3 & 6 \end{vmatrix} + 3\begin{vmatrix} -2 & -4 \\ 3 & 5 \end{vmatrix}$$

$$\det(A) = 1(-24 + 25) - 2(-12 + 15) + 3(-10 + 12) = 1 - 6 + 6 = 1$$

Since $\det(A) = 1 \neq 0$, the system has a unique solution.

3) **Cramer's Rule:**

$$\det(A_x) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -4 & -5 \\ 3 & 5 & 6 \end{vmatrix} = 13 \implies x = \frac{13}{1} = 13$$

$$\det(A_y) = \begin{vmatrix} 1 & 1 & 3 \\ -2 & 2 & -5 \\ 3 & 3 & 6 \end{vmatrix} = -12 \implies y = \frac{-12}{1} = -12$$

$$\det(A_z) = \begin{vmatrix} 1 & 2 & 1 \\ -2 & -4 & 2 \\ 3 & 5 & 3 \end{vmatrix} = 4 \implies z = \frac{4}{1} = 4$$

Solution: $(x, y, z) = (13, -12, 4)$.

Exercise 2 Solution (Matrix Inversion)

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 \\ 5 \\ 12 \end{pmatrix}$$

1) **Invertibility:** $\det(A) = 1(2 - 1) + 2(4 + 3) + 1(-2 - 3) = 1 + 14 - 5 = 10$. Since $\det(A) \neq 0$, A is invertible.

2) **Inverse Matrix:** $A^{-1} = \frac{1}{10}\begin{pmatrix} 1 & 3 & 1 \\ -7 & -1 & 3 \\ -5 & -5 & 5 \end{pmatrix}$

3) **Solution:** $\mathbf{x} = A^{-1}B = \frac{1}{10}\begin{pmatrix} 30 \\ 10 \\ 20 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$.

Solution: Exercise 3

System (S_3)**:** $\begin{cases} x + y + 2z = 3 \\ x + 2y + z = 1 \\ 2x + y + z = 0 \end{cases}$ [cite: 31]

Method 1: Cramer's Rule

1. Calculate the Main Determinant ($\det(A)$**):**

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

$$\det(A) = 1(2-1) - 1(1-2) + 2(1-4) = 1(1) + 1 + 2(-3) = -4$$

Since $\det(A) = -4 \neq 0$, the system has a unique solution.

2. Calculate Variable Determinants: Replace the columns of A with the constant vector $B = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ [cite: 31]:

$$\det(A_x) = \begin{vmatrix} \mathbf{3} & 1 & 2 \\ \mathbf{1} & 2 & 1 \\ \mathbf{0} & 1 & 1 \end{vmatrix} = 3(2-1) - 1(1-0) + 2(1-0) = 3 - 1 + 2 = 4$$

$$\det(A_y) = \begin{vmatrix} 1 & \mathbf{3} & 2 \\ 1 & \mathbf{1} & 1 \\ 2 & \mathbf{0} & 1 \end{vmatrix} = 1(1-0) - 3(1-2) + 2(0-2) = 1 + 3 - 4 = 0$$

$$\det(A_z) = \begin{vmatrix} 1 & 1 & \mathbf{3} \\ 1 & 2 & \mathbf{1} \\ 2 & 1 & \mathbf{0} \end{vmatrix} = 1(0-1) - 1(0-2) + 3(1-4) = -1 + 2 - 9 = -8$$

3. Final Values:

$$x = \frac{\det(A_x)}{\det(A)} = \frac{4}{-4} = -1, \quad y = \frac{0}{-4} = 0, \quad z = \frac{-8}{-4} = 2$$

Method 2: Inverse Matrix Method

The solution is given by $\mathbf{x} = A^{-1}B$.

1. Matrix of Cofactors (C):

$$C = \begin{pmatrix} +(2-1) & -(1-2) & +(1-4) \\ -(1-2) & +(1-4) & -(1-2) \\ +(1-4) & -(1-2) & +(2-1) \end{pmatrix} = \begin{pmatrix} 1 & 1 & -3 \\ 1 & -3 & 1 \\ -3 & 1 & 1 \end{pmatrix}$$

2. Adjugate and Inverse Matrix: Since C is symmetric in this specific case, $\text{Adj}(A) = C^T = C$.

$$A^{-1} = \frac{1}{-4} \begin{pmatrix} 1 & 1 & -3 \\ 1 & -3 & 1 \\ -3 & 1 & 1 \end{pmatrix}$$

3. Solving $\mathbf{x} = A^{-1}B$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} 1 & 1 & -3 \\ 1 & -3 & 1 \\ -3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} 3+1+0 \\ 3-3+0 \\ -9+1+0 \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} 4 \\ 0 \\ -8 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

Conclusion: Both methods yield the same solution: $(x, y, z) = (-1, 0, 2)$.