Chapter 4: Systems of Linear Equations

Mathematics Course

2026

1 Definitions and Interpretations

1.1 General Definition

A **system of linear equations** (or linear system) is a collection of m linear equations involving n unknowns $x_1, x_2, \ldots, x_n$. It is written in the general form:

$$(S) : \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

where:

- $a_{ij} \in \mathbb{K}$ are the **coefficients** of the system.
- $b_i \in \mathbb{K}$ are the **constants** (right-hand side).
- If all $b_i = 0$, the system is called **homogeneous**.

1.2 Matrix Representation

The system (S) can be represented by the matrix equation:

$$A\mathbf{x} = \mathbf{B}$$

where:

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Definition 1.1 (Consistency)**.** *A system is **consistent** if it has at least one solution. It is **inconsistent** if it has no solution.*

1.3 Geometric Interpretation

In $\mathbb{R}^n$, each linear equation represents a **hyperplane**.

- In $\mathbb{R}^2$: Lines. A solution is a point of intersection.
- In $\mathbb{R}^3$: Planes.

2 Theorems and Propositions

Proposition 2.1 (Structure of Solutions). *For any linear system over $\mathbb{R}$ or $\mathbb{C}$, exactly one of the following is true:*

1. *The system has a* ***unique solution****.*
2. *The system has* ***infinitely many solutions****.*
3. *The system has* ***no solution****.*

Theorem 2.1 (Rouché-Capelli Theorem). *A system $A\mathbf{x} = \mathbf{B}$ is consistent if and only if the rank of the coefficient matrix A is equal to the rank of the augmented matrix $[A|B]$:*

$$rank(A) = rank([A|B])$$

- *If $rank(A) = n$ (number of unknowns), the solution is unique.*
- *If $rank(A) < n$, there are infinitely many solutions.*

3 Cramer's Systems (Square Systems)

Definition 3.1 (Cramer's System). *A system is called a* ***Cramer's System*** *if:*

1. *The number of equations equals the number of unknowns ($m = n$).*
2. *The determinant of the coefficient matrix is non-zero:* $\det(A) \neq 0$.

Theorem 3.1 (Cramer's Rule). *If $\det(A) \neq 0$, the system $A\mathbf{x} = \mathbf{B}$ has a unique solution given by:*

$$x_j = \frac{\det(A_j)}{\det(A)}, \quad j = 1, \ldots, n$$

where A_j is the matrix obtained by replacing the j-th column of A with the vector $\mathbf{B}$.

4 Examples with Solutions

Example 4.1. *Solve the following 2×2 system using Cramer's Rule:*

$$\begin{cases} 3x - 2y = 7 \\ x + 4y = -7 \end{cases}$$

Solution: 1. Calculate the main determinant:

$$\det(A) = \begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix} = (3)(4) - (-2)(1) = 12 + 2 = 14$$

Since $\det(A) = 14 \neq 0$, the system is a Cramer system.
2. Calculate $\det(A_x)$ and $\det(A_y)$:

$$\det(A_x) = \begin{vmatrix} 7 & -2 \\ -7 & 4 \end{vmatrix} = (7)(4) - (-2)(-7) = 28 - 14 = 14$$

$$\det(A_y) = \begin{vmatrix} 3 & 7 \\ 1 & -7 \end{vmatrix} = (3)(-7) - (7)(1) = -21 - 7 = -28$$

3. Find x and y:

$$x = \frac{14}{14} = 1, \quad y = \frac{-28}{14} = -2$$

The solution is $(x, y) = (1, -2)$.

Example 4.2. *Solve the following* 3×3 *system representing three planes in* $\mathbb{R}^3$*:*

$$\begin{cases} 2x + y - z = 1 \\ x - y + z = 2 \\ x + 2y + z = 8 \end{cases}$$

Solution: 1. Extract the coefficient matrix A and constant vector B:

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix}$$

2. Calculate the determinant of A:

$$\det(A) = 2(-1 - 2) - 1(1 - 1) - 1(2 + 1) = -6 - 0 - 3 = -9$$

3. Calculate the determinants for each variable:

$$\det(A_x) = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 8 & 2 & 1 \end{vmatrix} = -9$$

$$\det(A_y) = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 8 & 1 \end{vmatrix} = -18$$

$$\det(A_z) = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & 8 \end{vmatrix} = -27$$

4. Final Solution:

$$x = \frac{-9}{-9} = 1, \quad y = \frac{-18}{-9} = 2, \quad z = \frac{-27}{-9} = 3$$

The solution set is $\{(1, 2, 3)\}$.