Chapter 9

Linear Equations

Linear algebra is an essential tool for all branches of mathematics, especially when it comes to modeling and then numerically solving problems from various fields: physical or mechanical sciences, life sciences, chemistry, economics, engineering sciences...

Linear equations, through their applications in many contexts, form the computational basis of linear algebra. They also allow the treatment of a large part of linear algebra theories in finite-dimensional spaces.

Therefore, we will devote this part to the topic of linear systems with an arbitrary number of equations or unknowns. We will study several methods for solving such systems with some numerical examples to explain the steps involved in each method.

9.1 Systems of Linear Equations

In all that follows in this chapter, we consider the commutative field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 9.1

We call a linear system with n equations and p unknowns, or a linear system with coefficients in the field $\mathbb{K}$, any system of equations of the form:

$$(S)\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1p}x_p = b_1 \\ a_{21}x_1 + a_{12}x_2 + \cdots + a_{2p}x_p = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{np}x_p = b_n \end{cases}$$

where for each $1 \leq i \leq n$ and $1 \leq j \leq p$, the coefficients a_{ij} and b_i are in $\mathbb{K}$. The vector:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix} \in \mathbb{K}^p$$

that satisfies all the equations of system S is called a solution to the system S.
The vector:

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{K}^n$$

is called the right-hand side (or constant term vector) of the linear system S.

The set

$$\mathcal{H}(S) = \{x \in \mathbb{K}^p \mid x \text{ is a solution of } S\}$$

is called the solution set of system (S).

9.1.1 Special cases

1) If $n = p$, then system S is called a square system.

2) If $b_1 = b_2 = \cdots = b_n = 0$, then we call S a homogeneous system. It is often denoted by S_0:

$$(S_0)\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1p}x_p = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2p}x_p = 0 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{np}x_p = 0 \end{cases}$$

This is the homogeneous system associated with the linear system S.

Definition 9.2

Two systems S_1 and S_2 are said to be equivalent if they have the same solution set, i.e.,

$$\mathcal{H}(S_1) = \mathcal{H}(S_2).$$

9.1.2 Matrix form of a linear system

Definition 9.3

Let n and p be two non-zero natural numbers. Consider the following linear system:

$$(S)\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1p}x_p = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2p}x_p = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{np}x_p = b_n \end{cases}$$

We set:

$$A := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}, \quad X := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}, \quad B := \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

The matrix A is called the coefficient matrix of the linear system (S), X is the solution vector, and B is the right-hand side vector. The system can then be written in matrix form as:

$$AX = B.$$

Explicitly:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

This is called the matrix form of the linear system (S).

9.2 Solving Linear Systems

9.2.1 Substitution method

To determine if a linear system has zero, one, or infinitely many solutions, and to find them, one method is substitution. For example, consider the following linear system:

$$(S)\left\{\begin{array}{rcl} 3x+2y &=& 1 \\ 2x-7y &=& -2 \end{array}\right.$$

We rewrite the first equation as $y = \frac{1}{2} - \frac{3}{2}x$. Substituting this expression for y into the second equation yields an equivalent system:

$$\left\{\begin{array}{rcl} y &=& \frac{1}{2} - \frac{3}{2}x \\ 2x - 7(\frac{1}{2} - \frac{3}{2}x) &=& -2 \end{array}\right.$$

The second equation now contains only the variable x, which can be solved:

$$\left\{\begin{array}{rcl} y &=& \frac{1}{2} - \frac{3}{2}x \\ (2+\frac{21}{2})x &=& -2+\frac{7}{2} \end{array}\right. \iff \left\{\begin{array}{rcl} y &=& \frac{1}{2} - \frac{3}{2}x \\ \frac{25}{2}x &=& \frac{3}{2} \end{array}\right. \iff \left\{\begin{array}{rcl} y &=& \frac{1}{2} - \frac{3}{2}x \\ x &=& \frac{3}{25} \end{array}\right.$$

Finally, substitute the value of x back into the first equation:

$$\left\{\begin{array}{rcl} y &=& \frac{1}{2} - \frac{3}{2}\cdot\frac{3}{25} = \frac{1}{2} - \frac{9}{50} = \frac{25-9}{50} = \frac{16}{50} = \frac{8}{25} \\ x &=& \frac{3}{25} \end{array}\right.$$

Hence, the system has a unique solution $(x, y) = \left(\frac{3}{25}, \frac{8}{25}\right)$. The solution set is:

$$\mathcal{H}(\mathcal{S}) = \left\{ \left(\frac{3}{25}, \frac{8}{25} \right) \right\}.$$

9.2.2 Cramer's rule

Consider a simple linear system of two equations in two unknowns:

$$\begin{cases} ax + by &= e \\ cx + dy &= f \end{cases}$$

Let Δ be the determinant of the coefficient matrix:

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

If $\Delta \neq 0$, the system has a unique solution given by Cramer's rule:

$$x = \frac{\Delta_x}{\Delta} = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{ad - bc}, \quad y = \frac{\Delta_y}{\Delta} = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{ad - bc}.$$

For x, the first column is replaced by the right-hand side; for y, the second column is replaced.

Example 9.1

Consider the system

$$\begin{cases} tx - 2y &= 1 \\ 3x + ty &= 1 \end{cases}$$

depending on the parameter $t \in \mathbb{R}$. The determinant of the system is:

$$\Delta = \begin{vmatrix} t & -2 \\ 3 & t \end{vmatrix} = t^2 + 6 \neq 0 \text{ for all real } t.$$

Thus, it is a Cramer system with a unique solution:

$$x = \frac{\begin{vmatrix} 1 & -2 \\ 1 & t \end{vmatrix}}{t^2 + 6} = \frac{t+2}{t^2+6}, \quad y = \frac{\begin{vmatrix} t & 1 \\ 3 & 1 \end{vmatrix}}{t^2 + 6} = \frac{t-3}{t^2+6}.$$

For each t, the solution set is:

$$\mathcal{H}(\mathcal{S}) = \left\{ \left(\frac{t+2}{t^2+6}, \frac{t-3}{t^2+6} \right) \right\}.$$

9.2.3 Gauss's method (Gaussian elimination)

Using elementary row operations on the augmented matrix $[A|B]$, Gauss's method is a systematic way to transform a linear system S into an equivalent system S' whose matrix is in row echelon form (upper triangular, but not necessarily with 1's on the diagonal). The goal is to create zeros below the main diagonal.

Before we start, we recall the elementary row operations that yield an equivalent system (one with the same solution set):

- Swapping two equations (rows).
- Multiplying both sides of an equation (a row) by a non-zero constant.
- Adding a multiple of one equation (row) to another equation (row).

The principle of Gauss's method is to transform the linear system into an equivalent triangular system, from which the variables can be easily found by back-substitution.

Performing the elimination

Assuming $a_{11} \neq 0$, we use it as a pivot to eliminate x_1 from equations 2 to n. For each row $i \geq 2$, we perform the operation $L_i \leftarrow L_i - \frac{a_{i1}}{a_{11}} L_1$. After this step, we obtain an equivalent system where x_1 only appears in the first equation. We then repeat the process on the smaller subsystem (rows 2 to n, columns 2 to p), using $a_{22}^{(1)}$ as the next pivot, provided it is non-zero. If the pivot is zero, we swap the current row with a lower row that has a non-zero entry in that column. If no such non-zero entry exists, the system may have no unique solution.

Example 9.2

Let's use Gauss's method to find the solutions of the system:

$$\begin{cases} x + y + 2z &= 3 \\ x + 2y + z &= 1 \\ 2x + y + z &= 0 \end{cases}$$

We write the augmented matrix and perform row operations:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right] \xrightarrow{L_2 \leftarrow L_2 - L_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{L_3 \leftarrow L_3 - 2L_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & -1 & -3 & -6 \end{array}\right]$$

$$\xrightarrow{L_3 \leftarrow L_3 + L_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & -4 & -8 \end{array}\right].$$

This corresponds to the triangular system:

$$\begin{aligned} x+y+2z &= 3 \\ y-z &= -2 \\ -4z &= -8 \end{aligned}$$

Solving by back-substitution: from the last equation, $z=2$. Substituting into the second: $y-2=-2 \implies y=0$. Substituting into the first: $x+0+2(2)=3 \implies x=-1$. So the unique solution is $(x,y,z)=(-1,0,2)$.

9.2.4 Matrix inverse method

A linear system in matrix form $AX=B$ can be solved by finding the inverse of A, if it exists. For a 2×2 system:

$$\begin{cases} ax+by &= e \\ cx+dy &= f \end{cases} \quad \text{is equivalent to} \quad AX=Y,$$

where $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $X=\begin{pmatrix} x \\ y \end{pmatrix}$, $Y=\begin{pmatrix} e \\ f \end{pmatrix}$. If $\det(A)=ad-bc\neq 0$, then A is invertible and

$$A^{-1}=\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The unique solution is $X=A^{-1}Y$.

Example 9.3

Solve the following linear system for the parameter $t\in\mathbb{R}$:

$$\begin{cases} x+y &= 1 \\ x+t^2y &= t \end{cases}$$

The determinant is $\Delta=\begin{vmatrix} 1 & 1 \\ 1 & t^2 \end{vmatrix}=t^2-1$.

(1) Case 1: $t\neq 1$ and $t\neq -1$ ($\Delta\neq 0$). The matrix $A=\begin{pmatrix} 1 & 1 \\ 1 & t^2 \end{pmatrix}$ is invertible, with inverse

$$A^{-1}=\frac{1}{t^2-1}\begin{pmatrix} t^2 & -1 \\ -1 & 1 \end{pmatrix}.$$

The solution is

$$X=A^{-1}Y=\frac{1}{t^2-1}\begin{pmatrix} t^2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ t \end{pmatrix}=\frac{1}{t^2-1}\begin{pmatrix} t^2-t \\ t-1 \end{pmatrix}=\begin{pmatrix} \frac{t}{t+1} \\ \frac{1}{t+1} \end{pmatrix}.$$

For each $t\neq\pm 1$, the solution set is $\mathcal{H}(\mathcal{S})=\left\{\left(\frac{t}{t+1},\frac{1}{t+1}\right)\right\}$.

(2) Case 2: $t = 1$. The system becomes:

$$\begin{cases} x+y &= 1 \\ x+y &= 1 \end{cases}$$

The two equations are identical. There are infinitely many solutions:

$$\mathcal{H}(\mathcal{S}) = \{(x, 1-x) \mid x \in \mathbb{R}\}.$$

(3) Case 3: $t = -1$. The system becomes:

$$\begin{cases} x+y &= 1 \\ x+y &= -1 \end{cases}$$

The equations are contradictory, so there are no solutions:

$$\mathcal{H}(\mathcal{S}) = \varnothing.$$

9.3 Exercise Series N° 7

Exercise 9.1

Solve the following linear systems using the Gauss method:

$$\begin{cases} x & +y & +2z & = & 3 \\ x & +2y & +z & = & 1 \\ 2x & +y & +z & = & 0 \end{cases}, \qquad \begin{cases} x & & +2z & = & 1 \\ & -y & +z & = & 2 \\ x & -2y & & = & 1 \end{cases}.$$

Solution

For the first system, using Gauss's method as shown in the example in the text, we obtain:

$$\iff \begin{cases} x &= -1 \\ y &= 0 \\ z &= 2 \end{cases}$$

The solution is $(x, y, z) = (-1, 0, 2)$.

For the second system, we proceed similarly:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & -1 & 1 & 2 \\ 1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{L_3 \leftarrow L_3 - L_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & -2 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{L_3 \leftarrow L_3 - 2L_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & -4 & -4 \end{array}\right].$$

This gives the system:

$$\begin{aligned} x + 2z &= 1 \\ -y + z &= 2 \\ -4z &= -4 \end{aligned}$$

Back-substituting: $z = 1$, then $-y + 1 = 2 \implies y = -1$, and $x + 2(1) = 1 \implies x = -1$. So the solution is $(x, y, z) = (-1, -1, 1)$.

Exercise 9.2

(1) Find the solutions to the following system in four different ways (by substitution, by Gauss's pivot method, by inverting the coefficient matrix, and by using Cramer's rule):

$$\begin{cases} 2x + y = 1 \\ 3x + 7y = -2 \end{cases}$$

(2) Choose the method that seems fastest for solving, depending on the values of a, the following systems:

$$\begin{cases} ax + y = 2 \\ (a^2+1)x + 2ay = 1 \end{cases} \qquad \begin{cases} (a+1)x + (a-1)y = 1 \\ (a-1)x + (a+1)y = 1 \end{cases}$$

Solution

(1.1) **Substitution method:**
From the first equation, $y = 1 - 2x$. Substituting into the second:

$$3x + 7(1-2x) = -2 \implies 3x + 7 - 14x = -2 \implies -11x = -9 \implies x = \frac{9}{11}.$$

Then $y = 1 - 2\frac{9}{11} = 1 - \frac{18}{11} = -\frac{7}{11}$. The solution is $(\frac{9}{11}, -\frac{7}{11})$.

(2.1) **Gauss's method:**
Write the augmented matrix and perform $L_2 \leftarrow 2L_2 - 3L_1$:

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 3 & 7 & -2 \end{array}\right] \xrightarrow{L_2 \leftarrow 2L_2 - 3L_1} \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 11 & -7 \end{array}\right].$$

From the second row: $11y = -7 \implies y = -\frac{7}{11}$. Substituting into the first: $2x - \frac{7}{11} = 1 \implies 2x = 1 + \frac{7}{11} = \frac{18}{11} \implies x = \frac{9}{11}$.

(3.1) **Matrix inverse method:**
The system is $AX = Y$ with $A = \begin{pmatrix} 2 & 1 \\ 3 & 7 \end{pmatrix}$, $Y = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. $\det A = 14 - 3 = 11 \neq 0$. The inverse is $A^{-1} = \frac{1}{11}\begin{pmatrix} 7 & -1 \\ -3 & 2 \end{pmatrix}$. The solution is $X = A^{-1}Y = \frac{1}{11}\begin{pmatrix} 7 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 7+2 \\ -3-4 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 9 \\ -7 \end{pmatrix} = \begin{pmatrix} 9/11 \\ -7/11 \end{pmatrix}$.

(4.1) **Cramer's rule:**
$\Delta = \begin{vmatrix} 2 & 1 \\ 3 & 7 \end{vmatrix} = 11$. $\Delta_x = \begin{vmatrix} 1 & 1 \\ -2 & 7 \end{vmatrix} = 7 - (-2) = 9$. $\Delta_y = \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} = -4 - 3 = -7$.
$x = \Delta_x/\Delta = 9/11$, $y = \Delta_y/\Delta = -7/11$.

(2) For the first system $\begin{cases} ax + y = 2 \\ (a^2+1)x + 2ay = 1 \end{cases}$, the determinant is:

$$\Delta = \begin{vmatrix} a & 1 \\ a^2+1 & 2a \end{vmatrix} = 2a^2 - (a^2+1) = a^2 - 1.$$

So there is a unique solution if $a \neq \pm 1$. Using Cramer's rule (or substitution) yields:

$$x = \frac{\begin{vmatrix} 2 & 1 \\ 1 & 2a \end{vmatrix}}{a^2-1} = \frac{4a-1}{a^2-1}, \quad y = \frac{\begin{vmatrix} a & 2 \\ a^2+1 & 1 \end{vmatrix}}{a^2-1} = \frac{a - 2(a^2+1)}{a^2-1} = \frac{a - 2a^2 - 2}{a^2-1} = \frac{-2a^2 + a - 2}{a^2-1}.$$

If $a = 1$, the system becomes

$$\begin{cases} x + y = 2 \\ 2x + 2y = 1 \end{cases}$$

which simplifies to $x + y = 2$ and $x + y = 1/2$, impossible. No solution. If $a = -1$, the system becomes

$$\begin{cases} -x + y = 2 \\ 2x - 2y = 1 \end{cases}$$

which gives $y = x+2$ from the first, and $2x - 2(x+2) = -4 = 1$ from the second, impossible. No solution.

For the second system

$$\begin{cases} (a+1)x + (a-1)y = 1 \\ (a-1)x + (a+1)y = 1 \end{cases}$$

the determinant is:

$$\Delta = \begin{vmatrix} a+1 & a-1 \\ a-1 & a+1 \end{vmatrix} = (a+1)^2 - (a-1)^2 = (a^2+2a+1) - (a^2-2a+1) = 4a.$$

If $a \neq 0$, there is a unique solution. By Cramer's rule:

$$x = \frac{\begin{vmatrix} 1 & a-1 \\ 1 & a+1 \end{vmatrix}}{4a} = \frac{(a+1)-(a-1)}{4a} = \frac{2}{4a} = \frac{1}{2a},$$

$$y = \frac{\begin{vmatrix} a+1 & 1 \\ a-1 & 1 \end{vmatrix}}{4a} = \frac{(a+1)-(a-1)}{4a} = \frac{2}{4a} = \frac{1}{2a}.$$

If $a = 0$, the system becomes $\begin{cases} x - y = 1 \\ -x + y = 1 \end{cases}$, i.e., $x - y = 1$ and $x - y = -1$, which is impossible. No solution.

Exercise 9.3

Find the solutions to the following system:

$$(S) = \begin{cases} 3x & & +2z & & = 0 \\ & 3y & +z & +3t & = 0 \\ x & +y & +z & +t & = 0 \\ 2x & -y & +z & -t & = 0 \end{cases}$$

Solution

We simplify the system. First, reorder the equations and variables for convenience. Let's use the third equation as the first and order variables as y, t, x, z:

$$\begin{cases} y + t + x + z = 0 & (L_1) \\ 3y + 3t + z = 0 & (L_2) \\ -y - t + 2x + z = 0 & (L_3) \\ 3x + 2z = 0 & (L_4) \end{cases}$$

Perform $L_2 \leftarrow L_2 - 3L_1$, $L_3 \leftarrow L_3 + L_1$:

$$\begin{cases} y + t + x + z = 0 \\ -3x - 2z = 0 \\ 3x + 2z = 0 \\ 3x + 2z = 0 \end{cases}$$

The last three equations are all equivalent to $3x + 2z = 0$. So the system reduces to:

$$\begin{cases} y + t + x + z = 0 \\ 3x + 2z = 0 \end{cases}$$

We can treat x and y as free parameters. From the second equation, $z = -\frac{3}{2}x$. Substituting into the first: $y + t + x - \frac{3}{2}x = 0 \implies y + t - \frac{1}{2}x = 0 \implies t = \frac{1}{2}x - y$. Thus, the solution set is:

$$\mathcal{H}(S) = \left\{ \left(x, y, -\frac{3}{2}x, \frac{1}{2}x - y \right) \mid x, y \in \mathbb{R} \right\}.$$

Exercise 9.4

Solve the following system:

$$\begin{cases} 3x - y + 2z = a \\ -x + 2y - 3z = b \\ x + 2y + z = c \end{cases}$$

Solution

Using Gauss's method, we perform $L_2 \leftarrow 3L_2 + L_1$ and $L_3 \leftarrow 3L_3 - L_1$:

$$\begin{cases} 3x - y + 2z = a \\ 5y - 7z = 3b + a \\ 7y + z = 3c - a \end{cases}$$

Then perform $L_3 \leftarrow 5L_3 - 7L_2$ to eliminate y from the third equation:

$$\begin{cases} 3x & - & y & + & 2z & = & a \\ & & 5y & - & 7z & = & 3b+a \\ & & & & 54z & = & 5(3c-a)-7(3b+a) = 15c-5a-21b-7a = 15c-21b-12a \end{cases}$$

From the last equation:

$$z = \frac{1}{54}(-12a-21b+15c) = \frac{1}{18}(-4a-7b+5c).$$

Then from the second equation:

$$\begin{aligned} 5y &= 3b+a+7z = 3b+a+7\cdot\frac{1}{18}(-4a-7b+5c) \\ &= \frac{54b+18a}{18} + \frac{-28a-49b+35c}{18} \\ &= \frac{18a-28a+54b-49b+35c}{18} = \frac{-10a+5b+35c}{18} = \frac{5(-2a+b+7c)}{18}. \end{aligned}$$

So $y = \frac{1}{5}\cdot\frac{5(-2a+b+7c)}{18} = \frac{-2a+b+7c}{18}$. Finally, from the first equation:

$$\begin{aligned} 3x &= a+y-2z = a+\frac{-2a+b+7c}{18} - 2\cdot\frac{-4a-7b+5c}{18} \\ &= \frac{18a}{18} + \frac{-2a+b+7c}{18} + \frac{8a+14b-10c}{18} \\ &= \frac{18a-2a+8a+b+14b+7c-10c}{18} = \frac{24a+15b-3c}{18} = \frac{3(8a+5b-c)}{18}. \end{aligned}$$

So $x = \frac{1}{3}\cdot\frac{3(8a+5b-c)}{18} = \frac{8a+5b-c}{18}$. Thus, the solution is:

$$\begin{cases} x &= \dfrac{8a+5b-c}{18}, \\ y &= \dfrac{-2a+b+7c}{18}, \\ z &= \dfrac{-4a-7b+5c}{18}. \end{cases}$$

Exercise 9.5

Solve the following systems using Cramer's rule:

$$1) \begin{cases} x+y+2z &= 3 \\ x+2y+z &= 1 \\ 2x+y+z &= 0 \end{cases} \qquad 2) \begin{cases} x+2z &= 1 \\ -y+z &= 2 \\ x-2y &= 1 \end{cases}$$

Solution

(1) Check if the system is a Cramer system by calculating its determinant:

$$\det(A) = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = -4 \neq 0.$$

Thus, it is a Cramer system. The solutions are:

$$x = \frac{\begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix}}{-4} = \frac{4}{-4} = -1,$$

$$y = \frac{\begin{vmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix}}{-4} = \frac{0}{-4} = 0,$$

$$z = \frac{\begin{vmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{vmatrix}}{-4} = \frac{-8}{-4} = 2.$$

(2) First, write the system in a standard form with all variables in order:

$$\begin{cases} x & & + & 2z & = & 1 \\ & -y & + & z & = & 2 \\ x & -2y & & & = & 1 \end{cases}$$

The coefficient matrix is $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}$. Its determinant is:

$$\det(A) = \begin{vmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{vmatrix} = 1\cdot(0\cdot 0 - 1\cdot(-2)) - 0 + 2\cdot(0\cdot(-2) - (-1)\cdot 1) = 2 + 2\cdot(0+1) = 4 \neq 0.$$

Thus, it is a Cramer system. The solutions are:

Let's recalculate carefully:

$$x = \frac{\begin{vmatrix} 1 & 0 & 2 \\ 2 & -1 & 1 \\ 1 & -2 & 0 \end{vmatrix}}{4} = \frac{-4}{4} = -1$$

$$y = \frac{\begin{vmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{vmatrix}}{4} = \frac{-4}{4} = -1.$$

$$z = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -2 & 1 \end{vmatrix}}{4} = \frac{4}{4} = 1.$$

So the solution is $(x, y, z) = (-1, -1, 1)$.

Exercise 9.6

Solve the following system using the inverse matrix method, and what is the geometric interpretation of the result you obtain?

$$\begin{cases} x + my & = & -3 \\ mx + 4y & = & 6 \end{cases}$$

Solution

The determinant of the coefficient matrix is:

$$\det \begin{pmatrix} 1 & m \\ m & 4 \end{pmatrix} = 4 - m^2.$$

Thus, det $= 0$ when $m = 2$ or $m = -2$.

- If $m = 2$, the system becomes $\begin{cases} x + 2y = -3 \\ 2x + 4y = 6 \end{cases}$. The second equation simplifies to $x + 2y = 3$, which contradicts the first. The system has no solution. Geometrically, this corresponds to two parallel lines.
- If $m = -2$, the system becomes $\begin{cases} x - 2y = -3 \\ -2x + 4y = 6 \end{cases}$. The second equation is -2 times the first, so it's the same line. The system has infinitely many solutions: $x = -3 + 2y$, y is free. Solution set: $\{(-3 + 2y, y) | y \in \mathbb{R}\}$. Geometrically, this corresponds to two coincident lines.
- If $m \neq 2$ and $m \neq -2$, the determinant is non-zero, so there is a unique solution. The inverse of $M = \begin{pmatrix} 1 & m \\ m & 4 \end{pmatrix}$ is:

  $$M^{-1} = \frac{1}{4 - m^2} \begin{pmatrix} 4 & -m \\ -m & 1 \end{pmatrix}.$$

  The solution is:

  $$X = M^{-1} \begin{pmatrix} -3 \\ 6 \end{pmatrix} = \frac{1}{4 - m^2} \begin{pmatrix} 4 & -m \\ -m & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 6 \end{pmatrix} = \frac{1}{4 - m^2} \begin{pmatrix} -12 - 6m \\ 3m + 6 \end{pmatrix} = \begin{pmatrix} \dfrac{-6(2+m)}{(2-m)(2+m)} \\ \dfrac{3(m+2)}{(2-m)(2+m)} \end{pmatrix} =$$

  So $x = \frac{-6}{2-m}$, $y = \frac{3}{2-m}$. Geometrically, for each $m \neq \pm 2$, the two lines intersect at this unique point.

Exercise 9.7

Discuss, depending on the value of the parameter $a \in \mathbb{R}$, the solutions of the system:

$$\begin{cases} 3x & +y & -z & = & 1 \\ x & -2y & +2z & = & a \\ x & +y & -z & = & 1 \end{cases}$$

Solution

The determinant of the coefficient matrix is:

$$\Delta = \begin{vmatrix} 3 & 1 & -1 \\ 1 & -2 & 2 \\ 1 & 1 & -1 \end{vmatrix} = 0,$$

since the first and third rows are not multiples, but let's check: $R_1 - 3R_3 = (0, -2, 2)$, which is $2R_2$. So the rows are linearly dependent, confirming $\Delta = 0$. This indicates either no solution or infinitely many.

Swap the first and third equations to simplify:

$$\begin{cases} x & +y & -z & = & 1 & (L_1) \\ x & -2y & +2z & = & a & (L_2) \\ 3x & +y & -z & = & 1 & (L_3) \end{cases}$$

Perform $L_2 \leftarrow L_2 - L_1$ and $L_3 \leftarrow L_3 - 3L_1$:

$$\begin{cases} x & +y & -z & = & 1 \\ & -3y & +3z & = & a-1 \\ & -2y & +2z & = & -2 \end{cases}$$

The last two equations can be simplified by dividing by common factors: $L_2/3$ and $L_3/2$ (though careful with scaling). It's clearer to rewrite them:

$$-3y + 3z = a - 1 \qquad \Rightarrow \quad y - z = \frac{1-a}{3}$$
$$-2y + 2z = -2 \qquad \Rightarrow \quad y - z = 1$$

For the system to be consistent, we require $\frac{1-a}{3} = 1$, which gives $1 - a = 3 \implies a = -2$.

- If $a = -2$, the last two equations are both $y - z = 1$. The system reduces to:

  $$\begin{cases} x + y - z = 1 \\ y - z = 1 \end{cases}$$

  From the second, $z = y - 1$. Substituting into the first: $x + y - (y - 1) = 1 \implies x + 1 = 1 \implies x = 0$. So $x = 0$, $z = y - 1$. The solution set is $\{(0, y, y-1) \mid y \in \mathbb{R}\}$, infinitely many solutions.

- If $a \neq -2$, the last two equations are contradictory ($y - z$ would have to equal two different numbers), so there is no solution.