Tutorial Series N°03
Full Solutions

Exercise 1

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -3 & 1 \\ -3 & 4 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 5 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 2 & -1 \\ 4 & 0 & 3 \end{pmatrix}.$$

1) **Type and size.**

$$A \in \mathbb{R}^{3\times 3} \text{ (square matrix of order 3)}, \qquad B \in \mathbb{R}^{3\times 2}, \qquad C \in \mathbb{R}^{2\times 3}.$$

2) **Compute (if possible).**

- $A + B$: not defined (sizes 3×3 and 3×2 are different).
- $3B$:
$$3B = \begin{pmatrix} 6 & -3 \\ 0 & 9 \\ 15 & 3 \end{pmatrix}.$$
- $3B + A$: not defined (sizes 3×2 and 3×3).
- $B + C$: not defined (sizes 3×2 and 2×3).
- AB (defined because $(3 \times 3)(3 \times 2) = (3 \times 2)$):
$$AB = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -3 & 1 \\ -3 & 4 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 9 & -10 \\ -11 & 14 \end{pmatrix}.$$
- $AI_3 = A$:
$$AI_3 = A.$$
- BC (defined because $(3 \times 2)(2 \times 3) = (3 \times 3)$):
$$BC = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 4 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 4 & -5 \\ 12 & 0 & 9 \\ 9 & 10 & -2 \end{pmatrix}.$$
- CB (defined because $(2 \times 3)(3 \times 2) = (2 \times 2)$):
$$CB = \begin{pmatrix} 1 & 2 & -1 \\ 4 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ 23 & -1 \end{pmatrix}.$$
- $3(AB)C$ (defined because (AB) is 3×2 and $(3 \times 2)(2 \times 3) = (3 \times 3)$):
$$(AB)C = \begin{pmatrix} 2 & -4 \\ 9 & -10 \\ -11 & 14 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 4 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -14 & 4 & -14 \\ -31 & 18 & -39 \\ 45 & -22 & 53 \end{pmatrix},$$
$$3(AB)C = \begin{pmatrix} -42 & 12 & -42 \\ -93 & 54 & -117 \\ 135 & -66 & 159 \end{pmatrix}.$$

- A^2:

$$A^2 = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -3 & 1 \\ -3 & 4 & -1 \end{pmatrix}^2 = \begin{pmatrix} -1 & 2 & -1 \\ -7 & 11 & -4 \\ 8 & -13 & 5 \end{pmatrix}.$$

3) **Transposes.**

$$A^\top = \begin{pmatrix} 1 & 2 & -3 \\ -1 & -3 & 4 \\ 0 & 1 & -1 \end{pmatrix}, \qquad B^\top = \begin{pmatrix} 2 & 0 & 5 \\ -1 & 3 & 1 \end{pmatrix}, \qquad C^\top = \begin{pmatrix} 1 & 4 \\ 2 & 0 \\ -1 & 3 \end{pmatrix}.$$

4) **Find a matrix D with no zero entries such that $AD = 0_{3\times3}$.**

First compute a nonzero vector in $\ker(A)$. Consider $u = (1,1,1)^\top$:

$$Au = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -3 & 1 \\ -3 & 4 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-1+0 \\ 2-3+1 \\ -3+4-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So $u \in \ker(A)$. If every column of D is a nonzero multiple of u, then each column is sent to 0 by A and hence $AD = 0$.

Choose for example (all entries are nonzero):

$$D = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

Then each column is a multiple of $(1,1,1)^\top$, so $AD = 0_{3\times3}$.

Exercise 2

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \end{pmatrix}.$$

1) **Determinants.**

For A: expand along the first row:

$$\det(A) = 2\det\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} - 1\det\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} + 0\cdot(\cdots).$$

$$\det\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} = 3\cdot1 - 0\cdot2 = 3, \qquad \det\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} = 0\cdot1 - 2\cdot1 = -2.$$

Thus

$$\det(A) = 2\cdot3 - 1\cdot(-2) = 6 + 2 = 8.$$

For B:

$$\det(B) = 4\cdot1 - 2\cdot1 = 2.$$

For C: note that row 2 is 2 times row 1:

$$(2,4,6) = 2(1,2,3) \;\Rightarrow\; \det(C) = 0.$$

2) **Traces.**

$$\text{tr}(A) = 2+3+1 = 6, \qquad \text{tr}(B) = 4+1 = 5, \qquad \text{tr}(C) = 1+4+1 = 6.$$

3) **Inverses (if possible).**

Matrix A: since $\det(A) = 8 \neq 0$, A is invertible. One possible form is

$$A^{-1} = \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \\ -\frac{3}{8} & \frac{1}{8} & \frac{3}{4} \end{pmatrix}.$$

Matrix B: since $\det(B) = 2 \neq 0$, B is invertible and

$$B^{-1} = \frac{1}{\det(B)} \begin{pmatrix} 1 & -1 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 2 \end{pmatrix}.$$

Matrix C: since $\det(C) = 0$, C is **not** invertible (no inverse exists).

Exercise 4

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \qquad f(x,y) = \big(2x+y,\ x-y,\ 3x+2y\big).$$

1) **Is f linear?**

Yes. Each component is a linear combination of x and y, and for all $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$,

$$f\big((x_1,y_1)+(x_2,y_2)\big) = f(x_1+x_2, y_1+y_2) = f(x_1,y_1) + f(x_2,y_2),$$

$$f(\lambda(x_1,y_1)) = f(\lambda x_1, \lambda y_1) = \lambda f(x_1,y_1).$$

2) **Matrix of f in the standard bases.**

Let $e_1 = (1,0)$ and $e_2 = (0,1)$ in $\mathbb{R}^2$. Then

$$f(e_1) = f(1,0) = (2,1,3), \qquad f(e_2) = f(0,1) = (1,-1,2).$$

Hence the matrix in the standard bases is

$$[f]_{\mathcal{B}_1,\mathcal{B}_2} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 2 \end{pmatrix}.$$

3) **Matrix of f in the bases $\mathcal{K}_1$ and $\mathcal{K}_2$.**

$$\mathcal{K}_1 = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} = \{u_1, u_2\}, \qquad \mathcal{K}_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} = \{w_1, w_2, w_3\}.$$

First compute:

$$f(u_1) = f(1,-1) = (1,2,1), \qquad f(u_2) = f(2,1) = (5,1,8).$$

Now express $f(u_1)$ and $f(u_2)$ in the basis $\{w_1, w_2, w_3\}$.
Any vector $v \in \mathbb{R}^3$ can be written as $v = \alpha w_1 + \beta w_2 + \gamma w_3$. Since

$$w_1 = (1,0,1), \quad w_2 = (0,1,1), \quad w_3 = (1,1,0),$$

we have

$$\alpha w_1 + \beta w_2 + \gamma w_3 = (\alpha+\gamma,\ \beta+\gamma,\ \alpha+\beta).$$

For $f(u_1) = (1,2,1)$:

$$(\alpha+\gamma,\ \beta+\gamma,\ \alpha+\beta) = (1,2,1) \Rightarrow \begin{cases} \alpha+\gamma = 1 \\ \beta+\gamma = 2 \\ \alpha+\beta = 1 \end{cases} \Rightarrow (\alpha,\beta,\gamma) = (0,1,1).$$

So $[f(u_1)]_{\mathcal{K}_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

For $f(u_2) = (5,1,8)$:

$$(\alpha+\gamma,\ \beta+\gamma,\ \alpha+\beta) = (5,1,8) \Rightarrow \begin{cases} \alpha+\gamma = 5 \\ \beta+\gamma = 1 \\ \alpha+\beta = 8 \end{cases} \Rightarrow (\alpha,\beta,\gamma) = (6,2,-1).$$

So $[f(u_2)]_{\mathcal{K}_2} = \begin{pmatrix} 6 \\ 2 \\ -1 \end{pmatrix}$.

Therefore, the matrix of f in $(\mathcal{K}_1, \mathcal{K}_2)$ is obtained by taking these coordinate vectors as columns:

$$[f]_{\mathcal{K}_1,\mathcal{K}_2} = \begin{pmatrix} 0 & 6 \\ 1 & 2 \\ 1 & -1 \end{pmatrix}.$$

Exercise 5

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \qquad f(x,y,z) = \big(x+y+z,\ 2y+z,\ 3z\big).$$

1) **f is an endomorphism.**
For $u, v \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$,

$$f(u+v) = f(u) + f(v), \qquad f(\lambda u) = \lambda f(u),$$

because each component of $f(x,y,z)$ is linear in x, y, z. Hence f is linear and maps $\mathbb{R}^3$ to $\mathbb{R}^3$, so it is an endomorphism.

2) **Matrix $A = M(f, \mathcal{B})$ in the standard basis $\mathcal{B} = (e_1, e_2, e_3)$.**
Compute columns $f(e_1), f(e_2), f(e_3)$:

$$f(1,0,0) = (1,0,0), \quad f(0,1,0) = (1,2,0), \quad f(0,0,1) = (1,1,3).$$

Therefore,

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

3) **Compute $f(v_1), f(v_2), f(v_3)$ in terms of (v_1, v_2, v_3).**

Let
$$v_1 = (1,0,0), \quad v_2 = (1,1,0), \quad v_3 = (1,1,1).$$
Compute:
$$f(v_1) = f(1,0,0) = (1,0,0) = 1 \cdot v_1,$$
$$f(v_2) = f(1,1,0) = (2,2,0) = 2 \cdot (1,1,0) = 2\,v_2,$$
$$f(v_3) = f(1,1,1) = (3,3,3) = 3 \cdot (1,1,1) = 3\,v_3.$$
So, in the basis $\mathcal{S} = (v_1, v_2, v_3)$:
$$[f(v_1)]_{\mathcal{S}} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad [f(v_2)]_{\mathcal{S}} = \begin{pmatrix}0\\2\\0\end{pmatrix}, \quad [f(v_3)]_{\mathcal{S}} = \begin{pmatrix}0\\0\\3\end{pmatrix}.$$

4) **Deduce that $\mathcal{S} = (v_1, v_2, v_3)$ is a basis of $\mathbb{R}^3$.**

Form the matrix having v_1, v_2, v_3 as columns:
$$G = \begin{pmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}.$$
It is upper triangular with diagonal entries $1,1,1$, so $\det(G) = 1 \neq 0$. Hence v_1, v_2, v_3 are linearly independent and $\mathcal{S}$ is a basis of $\mathbb{R}^3$.

5) **Matrix $D = M(f, \mathcal{S})$.**

Since $f(v_1) = 1v_1$, $f(v_2) = 2v_2$, $f(v_3) = 3v_3$, the matrix in $\mathcal{S}$ is diagonal:
$$D = \begin{pmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{pmatrix}.$$

6) **Passage matrix G from $\mathcal{S}$ to $\mathcal{B}$.**

The passage matrix from $\mathcal{S}$ to the standard basis $\mathcal{B}$ is exactly the matrix whose columns are v_1, v_2, v_3 written in the standard coordinates:
$$G = \begin{pmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}.$$
(So for any vector x, one has $[x]_{\mathcal{B}} = G[x]_{\mathcal{S}}$.)