Chapter 3: Matrices, Associated Matrices, and Determinants

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1 1. Definitions and Special Matrices

1.1 Definition of a Matrix

Definition 1.1. Let $\mathbb{K}$ be a field (usually $\mathbb{R}$ or $\mathbb{C}$). A **matrix** A of size $m \times n$ is a rectangular array of scalars from $\mathbb{K}$ arranged in m rows and n columns.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

The set of all $m \times n$ matrices is denoted $\mathcal{M}_{m,n}(\mathbb{K})$. If $m = n$, it is a **square matrix** $\mathcal{M}_n(\mathbb{K})$.

1.2 Special Matrices

Type	**Description**	**Example (3x3)**
Identity (I_n)	1 on diagonal, 0 elsewhere	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
Diagonal	$a_{ij} = 0$ for $i \neq j$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
Upper Triangular	$a_{ij} = 0$ for $i > j$	$\begin{pmatrix} 1 & 4 & 7 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{pmatrix}$
Lower Triangular	$a_{ij} = 0$ for $i < j$	$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 0 \\ 7 & 2 & 6 \end{pmatrix}$
Symmetric	$A = A^T$	$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 9 \\ 5 & 9 & 7 \end{pmatrix}$

2 2. Operations on Matrices

The set $\mathcal{M}_{m,n}(\mathbb{K})$ forms a vector space under standard addition and scalar multiplication.

2.1 Matrix Addition

Definition 2.1. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices of the **same size** $m \times n$. The sum $C = A + B$ is the matrix where each entry is the sum of the corresponding entries:

$$c_{ij} = a_{ij} + b_{ij}$$

Example 2.1 (2x2 Addition).

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

2.2 Scalar Multiplication

Definition 2.2. Let $A = (a_{ij})$ be a matrix and $\lambda \in \mathbb{K}$ be a scalar. The product λA is obtained by multiplying **every** entry of A by λ.

$$\lambda A = (\lambda a_{ij})$$

Example 2.2 (Scalar Multiplication). Let $\lambda = 2$ and $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 5 & 3 \end{pmatrix}$.

$$2A = \begin{pmatrix} 2(1) & 2(-1) & 2(0) \\ 2(2) & 2(5) & 2(3) \end{pmatrix} = \begin{pmatrix} 2 & -2 & 0 \\ 4 & 10 & 6 \end{pmatrix}$$

2.3 Transpose of a Matrix

Definition 2.3. The transpose of a matrix $A \in \mathcal{M}_{m,n}$, denoted A^T (or ${}^tA$), is the matrix in $\mathcal{M}_{n,m}$ obtained by swapping rows and columns.

$$(A^T)_{ij} = A_{ji}$$

Example 2.3 (Transpose). Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$.

$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

2.4 Matrix Multiplication

If $A \in \mathcal{M}_{m,n}$ and $B \in \mathcal{M}_{n,p}$, the product $C = AB$ is defined by $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.

Remark 2.1. Matrix multiplication is **not commutative**. In general, $AB \neq BA$.

2.4.1 Example: 2x2 Multiplication

Example 2.4. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$.

$$AB = \begin{pmatrix} 1(2) + 2(1) & 1(0) + 2(2) \\ 3(2) + 4(1) & 3(0) + 4(2) \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 10 & 8 \end{pmatrix}$$

2.4.2 Example: 3x3 Multiplication

Example 2.5. Let $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

$$AB = \begin{pmatrix} 3+0+2 & 1+0+0 & 0+0+2 \\ -3+0+0 & -1+2+0 & 0+1+0 \\ 0+0+1 & 0+0+0 & 0+0+1 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 2 \\ -3 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

3 3. Determinants

3.1 Definition

The determinant is a scalar value that can be computed from the elements of a square matrix. Geometrically, it represents the scaling factor of the volume transformation defined by the matrix.

Definition 3.1 (General Definition via Laplace Expansion)**.** Let $A = (a_{ij})$ be a square matrix of order n. The determinant of A, denoted $\det(A)$ or $|A|$, is defined recursively. For $n = 1$, $\det(A) = a_{11}$. For $n > 1$, expanding along the first row:

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(M_{1j})$$

where M_{1j} is the submatrix obtained by deleting the 1st row and j-th column.

3.2 Properties

The determinant satisfies several key properties that facilitate calculation.

Proposition 3.1 (Fundamental Properties)**.** *Let* $A, B \in \mathcal{M}_n(\mathbb{K})$.

1. $\det(I_n) = 1$.
2. ***Multiplicativity:*** $\det(AB) = \det(A)\det(B)$.
3. ***Transpose:*** $\det(A^T) = \det(A)$.
4. ***Invertibility:*** *A is invertible if and only if* $\det(A) \neq 0$.

Proposition 3.2 (Effect of Row Operations)**.**

1. *If we swap two rows (or columns), the determinant changes sign:* $\det(A') = -\det(A)$.
2. *If we multiply a row by a scalar λ, the determinant is multiplied by λ.*
3. *If we add a multiple of one row to another ($R_i \leftarrow R_i + \lambda R_j$), the determinant* ***remains unchanged****.*
4. *If a matrix has two identical rows (or columns),* $\det(A) = 0$.
5. *If a matrix has a row (or column) of zeros,* $\det(A) = 0$.

3.3 Calculation Methods

3.3.1 Order 2

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

3.3.2 Order 3: Rule of Sarrus

Remark 3.1. The Rule of Sarrus is a mnemonic strictly for $n = 3$.

Example 3.1 (Sarrus Method)**.** Compute $\det(A)$ for $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{pmatrix}$.

1. Downward diagonals (+): $(1 \cdot 4 \cdot 6) + (2 \cdot 5 \cdot 1) + (3 \cdot 0 \cdot 0) = 24 + 10 + 0 = 34$.
2. Upward diagonals (-): $(1 \cdot 4 \cdot 3) + (0 \cdot 5 \cdot 1) + (6 \cdot 0 \cdot 2) = 12 + 0 + 0 = 12$.

$$\det(A) = 34 - 12 = 22$$

3.3.3 General Case: Laplace Expansion

For any size $n \geq 3$, we can expand along any row i or column j.

Theorem 3.1.

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij}) \quad \textit{(Expansion along row i)}$$

Example 3.2 (Laplace Expansion)**.** Let $B = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 4 & 2 \\ 5 & 1 & 0 \end{pmatrix}$. Expand along Column 1 (since it has a zero):

$$\begin{aligned} \det(B) &= (+2)\begin{vmatrix} 4 & 2 \\ 1 & 0 \end{vmatrix} - (0)\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} + (5)\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} \\ &= 2(0 - 2) - 0 + 5(6 - 4) \\ &= 2(-2) + 5(2) = -4 + 10 = 6 \end{aligned}$$

4 4. Invertible Matrices

4.1 Definition and Uniqueness

Definition 4.1. A square matrix $A \in \mathcal{M}_n(\mathbb{K})$ is said to be **invertible** (or non-singular) if there exists a matrix $B \in \mathcal{M}_n(\mathbb{K})$ such that:

$$AB = BA = I_n$$

If such a matrix B exists, it is unique and is denoted by A^{-1}.

4.2 Properties of the Inverse

Proposition 4.1. *Let A and B be invertible matrices.*

1. $(A^{-1})^{-1} = A$.
2. $(AB)^{-1} = B^{-1}A^{-1}$ *(Note the reversal of order).*
3. $(A^T)^{-1} = (A^{-1})^T$.
4. $\det(A^{-1}) = \frac{1}{\det(A)}$.

4.3 Condition for Invertibility

Theorem 4.1. *A matrix A is invertible if and only if its determinant is non-zero:*

$$\det(A) \neq 0$$

4.4 Calculation Formula (Adjugate Method)

For any size n, the inverse can be calculated using the cofactor matrix.

$$A^{-1} = \frac{1}{\det(A)}(\text{Com}(A))^T$$

where $(\text{Com}(A))^T$ is called the **Adjugate Matrix**, denoted $\text{adj}(A)$. The entry (i, j) of the cofactor matrix is $C_{ij} = (-1)^{i+j} \det(M_{ij})$.

4.5 Example: 2x2 Inverse

For order 2, the formula simplifies: swap diagonal elements, negate off-diagonal elements.

Example 4.1. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$.

1. Calculate determinant: $\det(A) = (1)(5) - (2)(3) = 5 - 6 = -1$.
2. Apply formula:

$$A^{-1} = \frac{1}{-1}\begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$$

4.6 Example: 3x3 Inverse (Detailed Step-by-Step)

Example 4.2. Find the inverse of $B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}$.

Step 1: Check Determinant Using Row 1:

$$\det(B) = 1(0 - 24) - 2(0 - 20) + 3(0 - 5) = -24 + 40 - 15 = 1$$

Since $\det(B) \neq 0$, the inverse exists.

Step 2: Calculate Cofactors (C_{ij}) We calculate the minor for each position and apply the sign $(-1)^{i+j}$.

- $C_{11} = +(1(0) - 4(6)) = -24$

- $C_{12} = -(0(0) - 4(5)) = -(-20) = 20$
- $C_{13} = +(0(6) - 1(5)) = -5$
- $C_{21} = -(2(0) - 3(6)) = -(-18) = 18$
- $C_{22} = +(1(0) - 3(5)) = -15$
- $C_{23} = -(1(6) - 2(5)) = -(6 - 10) = 4$
- $C_{31} = +(2(4) - 3(1)) = 5$
- $C_{32} = -(1(4) - 3(0)) = -4$
- $C_{33} = +(1(1) - 2(0)) = 1$

The Cofactor Matrix is:

$$\text{Com}(B) = \begin{pmatrix} -24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & -4 & 1 \end{pmatrix}$$

Step 3: Transpose to get Adjugate Swap rows with columns:

$$\text{adj}(B) = (\text{Com}(B))^T = \begin{pmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{pmatrix}$$

Step 4: Final Formula

$$B^{-1} = \frac{1}{\det(B)}\text{adj}(B) = \frac{1}{1}\begin{pmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{pmatrix}$$

5 5. Matrix Representation of a Linear Application

5.1 Definition

Let $f : E \to F$ be a linear map. Let $\mathcal{B} = \{e_j\}$ be a basis of E and $\mathcal{C} = \{\epsilon_i\}$ be a basis of F. The matrix of f, denoted $M = \text{Mat}_{\mathcal{B},\mathcal{C}}(f)$, is constructed by taking the images of the basis vectors $f(e_j)$ and writing their coordinates as columns.

$$f(e_j) = \sum_{i=1}^{m} a_{ij}\epsilon_i$$

Example 5.1 (3x3 Linear Map). Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(x, y, z) = (2x - y,\ x + z,\ y + 3z)$. Canonical basis $\mathcal{B} = \{e_1, e_2, e_3\}$.

- $f(1, 0, 0) = (2, 1, 0) \implies$ Col 1: $(2, 1, 0)^T$
- $f(0, 1, 0) = (-1, 0, 1) \implies$ Col 2: $(-1, 0, 1)^T$
- $f(0, 0, 1) = (0, 1, 3) \implies$ Col 3: $(0, 1, 3)^T$

$$\text{Mat}(f) = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

6 6. Correspondence: Linear Maps and Matrices

There is a structural isomorphism between linear maps and matrices.

Linear Map Operation	**Matrix Operation**
$f(x)$	AX
$f+g$	$A+B$
λf	λA
Composition $g \circ f$	Product BA

Remark 6.1. Note the reversal in composition: $g \circ f$ means apply f first, then g. In matrices, we calculate BA because $(BA)X = B(AX)$.

7 7. Change of Basis Matrix (Passage Matrix)

7.1 Definition

Definition 7.1. Let E be a vector space of dimension n. Let $\mathcal{B} = \{e_1, \dots, e_n\}$ be the "old" basis and $\mathcal{B}' = \{u_1, \dots, u_n\}$ be the "new" basis.

The **change of basis matrix** (or transition matrix), denoted $P_{\mathcal{B}\to\mathcal{B}'}$ (or simply P), is the square matrix of order n whose j-th column is formed by the coordinates of the new basis vector u_j expressed in the old basis $\mathcal{B}$.

$$P = \begin{pmatrix} | & | & & | \\ u_1 & u_2 & \dots & u_n \\ | & | & & | \end{pmatrix}_{\text{in basis } \mathcal{B}}$$

7.2 Coordinate Change Formula

Proposition 7.1. *Let X be the column vector of coordinates of a vector v in the old basis $\mathcal{B}$. Let X' be the column vector of coordinates of the* **same** *vector v in the new basis $\mathcal{B}'$. Then:*

$$X = PX' \quad \Longleftrightarrow \quad X' = P^{-1}X$$

7.3 Example 1: 2x2 Change of Basis

Example 7.1. Let $\mathcal{B} = (e_1, e_2)$ be the canonical basis of $\mathbb{R}^2$. Consider a new basis $\mathcal{B}' = (u_1, u_2)$ where:

$$u_1 = (2, 1) = 2e_1 + 1e_2$$

$$u_2 = (-1, 1) = -1e_1 + 1e_2$$

The transition matrix P is formed by placing these vectors as columns:

$$P = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

7.4 Example 2: 3x3 Change of Basis

Example 7.2. Let $\mathcal{B} = (e_1, e_2, e_3)$ be the standard basis. Let the new basis $\mathcal{B}'$ be:

$$u_1 = (1, 1, 0), \quad u_2 = (1, 0, 1), \quad u_3 = (0, 1, 1)$$

To build P, we simply write u_1 in column 1, u_2 in column 2, etc.

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

8 8. Effect of Change of Basis on Matrix

How does the matrix of a linear operator change when we switch bases?

Theorem 8.1. *Let $f : E \to E$ be an endomorphism (linear operator).*

- *Let $A = Mat_{\mathcal{B}}(f)$ be the matrix in the old basis.*
- *Let $B = Mat_{\mathcal{B}'}(f)$ be the matrix in the new basis.*
- *Let P be the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$.*

Then the matrices are related by the similarity formula:

$$B = P^{-1}AP$$

Remark 8.1. The matrices A and B are called **similar**. They represent the same linear map from different viewpoints. They share the same:

- Determinant: $\det(B) = \det(A)$
- Trace: $\mathrm{tr}(B) = \mathrm{tr}(A)$
- Eigenvalues

8.1 Detailed 3x3 Example

Example 8.1. Let f be an operator with matrix A in the standard basis:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

(This represents a scaling of axes by 1, 3, and 5).

Let us change to the new basis $\mathcal{B}'$ from the previous section, where $P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. We want to find B, the matrix of f in this new basis.

Step 1: Calculate P^{-1}. First, $\det(P) = 1(0-1) - 1(1-0) + 0 = -2$. Using the cofactor method (omitted for brevity here, but standard calculation):

$$P^{-1} = \frac{1}{-2} \begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 \end{pmatrix}$$

Step 2: Compute the product AP.

$$AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 5 & 5 \end{pmatrix}$$

Step 3: Compute final product $B = P^{-1}(AP)$.

$$B = \begin{pmatrix} 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 5 & 5 \end{pmatrix}$$

Row 1: $0.5(1) + 0.5(3) - 0.5(0) = 2; \quad 0.5(1) + 0.5(0) - 0.5(5) = -2; \quad ...$

$$B = \begin{pmatrix} 2 & -2 & 1 \\ -1 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

Verification: Trace of A = $1 + 3 + 5 = 9$. Trace of B = $2 + 3 + 4 = 9$. The traces match!