

# Final Exam

**Exercise 1 (6pts=3pts+3pts)** Compute the following integrals:

$$\int \frac{x^2 + 1}{x(x-1)(x+1)} dx \quad \text{and} \quad \int \ln(x^2 + x + 1) dx.$$

**Exercise 2 (4pts=1pt+2pts+1pt)** Consider the real function  $f$  on  $D_f$  defined by:

$$f(x) = \sqrt{a^2 - x^2}, \quad \text{with } a > 0.$$

1. Give the domain  $D_f$ .
2. Calculate the value of the area delimited by the curve of  $f$  and the axis  $y = 0$  over the domain  $D_f$ .
3. Deduce the expression for the surface delimited by a circle with center  $(0, 0)$  and radius  $R$ .

We give:  $\sin(1) = -\sin(-1) = \frac{\pi}{2}$ .

**Exercise 3 (6pts=4pts+2pts)** Let us consider the following linear differential equation:

$$y' - \frac{y}{(x+1)^n} = \frac{1}{(x+1)^n}, \quad \text{with } x, y \in \mathbb{R}_+ \text{ and } n \in \mathbb{R}. \quad (1)$$

1. Solve the equation (2) according to the values of  $n$ .
2. Determine the solution of (2) that satisfies the condition  $y(0) = 0$  when  $n = 1$  and  $n = 2$ .

**Exercise 4 (4pts)** Find the global solution of the following differential equation

$$y'' + 2y' - 3y = 3x + 4 - 4e^{-3x} + 5 \sin(2x + 1).$$

# Final Exam Solution

## Solution of the Exercise 1

1. To compute the antiderivative we use the decomposition approach.

$$\int \frac{x^2 + 1}{x(x-1)(x+1)} dx = \int \frac{a}{x} + \frac{b}{x-1} + \frac{c}{x+1} dx$$

We have

- $\frac{x^2+1}{(x-1)(x+1)} = a + \frac{bx}{x-1} + \frac{cx}{x+1}$  for  $x = 0$  we get  $a = -1$ .
- $\frac{x^2+1}{x(x+1)} = \frac{a(x-1)}{x} + b + \frac{c(x-1)}{x+1}$  for  $x = 1$  we get  $b = 1$ .
- $\frac{x^2+1}{x(x-1)(x+1)} = \frac{a(x+1)}{x} + \frac{b(x+1)}{x-1} + c$  for  $x = -1$  we get  $c = 1$ .

$$\begin{aligned} \int \frac{x^2 + 1}{x(x-1)(x+1)} dx &= \int \frac{-1}{x} + \frac{1}{x-1} + \frac{1}{x+1} dx \\ &= -\ln(|x|) + \ln(|(x-1)|) + \ln(|(x+1)|) + c. \\ &= \ln\left(\left|\frac{x^2-1}{x}\right|\right) + c. \end{aligned}$$

2. To compute the considered integral we use the integration by parts.

$$\int \ln(x^2 + x + 1) dx = \int 1 \times \ln(x^2 + x + 1) dx.$$

we take

$$\begin{cases} u' = 1 \\ v = \ln(x^2 + x + 1) \end{cases} \implies \begin{cases} u = x \\ v' = \frac{2x+1}{x^2+x+1} \end{cases}$$

$$\begin{aligned} \int \ln(x^2 + x + 1) dx &= x \ln(x^2 + x + 1) - \int \frac{2x^2 + x}{x^2 + x + 1} dx. \\ &= x \ln(x^2 + x + 1) - \int 2 - \frac{x+2}{x^2 + x + 1} dx. \\ &= x \ln(x^2 + x + 1) - \int 2 dx - \frac{1}{2} \int \frac{2x+1}{x^2 + x + 1} dx - \frac{3}{2} \int \frac{1}{x^2 + x + 1} dx. \\ &= x \ln(x^2 + x + 1) - \int 2 dx - \frac{1}{2} \int \frac{2x+1}{x^2 + x + 1} dx - \frac{3}{2} \int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx. \\ &= x \ln(x^2 + x + 1) - 2x + \ln(x^2 + x + 1) - \frac{3}{2} \frac{2}{\sqrt{3}} \arctan\left(\frac{x + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c. \end{aligned}$$

**Solution of the Exercise 2**

1. The domain  $D_f$  is defined as follows:

$$D_f = \{x \in \mathbb{R} : a^2 - x^2 \geq 0\} = [-a; a].$$

2. The area delimited by the curve of  $f$  and the axis  $y = 0$  over the entire domain  $D_f$  is:

$$S = \int_{D_f} |f| dx = \int_{-a}^a \sqrt{a^2 - x^2} dx = \int_{-a}^a \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} dx.$$

We have

$$\begin{aligned} \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} dx &= (\alpha x + \beta) \times (\sqrt{a^2 - x^2}) + \lambda \int \frac{1}{\sqrt{a^2 - x^2}} dx \\ \implies \left[ \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} dx \right]' &= \left[ (\alpha x + \beta) \sqrt{a^2 - x^2} + \int \frac{\lambda}{\sqrt{a^2 - x^2}} dx \right]' \\ \implies \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} &= \frac{-2\alpha x^2 - \beta x + \lambda + \alpha^2 a^2}{\sqrt{a^2 - x^2}} \\ \implies \begin{cases} \alpha &= \frac{1}{2} \\ \beta &= 0 \\ \lambda &= \frac{a^2}{2} \end{cases}, \end{aligned}$$

thus,

$$\begin{aligned} \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} dx &= \frac{1}{2} \left( x \sqrt{a^2 - x^2} + a^2 \int \frac{1}{\sqrt{a^2 - x^2}} dx \right) \\ &= \frac{1}{2} \left( x \sqrt{a^2 - x^2} + a^2 \arcsin \left( \frac{x}{a} \right) \right) + c \end{aligned}$$

Consequently,

$$S = \int_{D_f} \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} dx = \frac{1}{2} \left( x \sqrt{a^2 - x^2} + a^2 \arcsin \left( \frac{x}{a} \right) \right) + c \Big|_{-a}^a = a^2 \pi.$$

It's to be noted that the integral in question can be also computed using the integration by parts. Indeed, we have:

$$\begin{aligned} I(x) &= \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} dx = \int \frac{a^2}{\sqrt{a^2 - x^2}} + \int \frac{-x^2}{\sqrt{a^2 - x^2}} \\ &= a^2 \arcsin \left( \frac{x}{a} \right) + \int x \times \frac{-2x}{2\sqrt{a^2 - x^2}} dx. \end{aligned}$$

To compute the last integral we take

$$\begin{cases} u &= x \\ v' &= \frac{-2x}{2\sqrt{a^2 - x^2}} \end{cases} \implies \begin{cases} u' &= 1 \\ v &= \sqrt{a^2 - x^2} \end{cases}$$

So,

$$\begin{aligned} I(x) &= a^2 \arcsin \left( \frac{x}{a} \right) + x \sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2} \\ &= a^2 \arcsin \left( \frac{x}{a} \right) + x \sqrt{a^2 - x^2} - I(x) \\ &= \frac{a^2}{2} \arcsin \left( \frac{x}{a} \right) + \frac{x}{2} \sqrt{a^2 - x^2} \end{aligned}$$

Consequently,

$$S = I(a) - I(-a) = a^2\pi.$$

3. Recall that the equation of a circle with center  $(0, 0)$  and radius  $R$  is given by

$$\begin{aligned} x^2 + y^2 = R^2 &\implies y = \pm\sqrt{R^2 - x^2} \\ \implies S &= \int_{-R}^R |\sqrt{a^2 - x^2}| dx + \int_{-R}^R |-\sqrt{a^2 - x^2}| dx = 2 \int_{-R}^R \sqrt{a^2 - x^2} dx = 2R^2\pi. \end{aligned}$$

### Solution of the Exercise 3

$$y' - \frac{y}{(x+1)^n} = \frac{1}{(x+1)^n}, \quad \text{with } x, y \in \mathbb{R}_+ \text{ and } n \in \mathbb{R}. \quad (2)$$

At first glance, the equation (2) is a linear differential equation with a right-hand side, but a little articulation can transform it into a linear differential equation without a right-hand side.

$$\begin{aligned} y' - \frac{y}{(x+1)^n} = \frac{1}{(x+1)^n} &\implies y' - \frac{y}{(x+1)^n} - \frac{1}{(x+1)^n} = 0 \\ &\implies y' - \frac{y+1}{(x+1)^n} = 0 \\ &\implies \frac{y'}{y+1} = \frac{1}{(x+1)^n} \\ &\implies \int \frac{1}{y+1} dy = \int \frac{1}{(x+1)^n} dx. \end{aligned}$$

At this level, we distinguish two possible cases, namely the case  $n = 1$  and the case  $n \neq 1$ .

**Case  $n = 1$**  In the case we get

$$\begin{aligned} \int \frac{1}{y+1} dy = \int \frac{1}{(x+1)} dx &\implies \ln(y+1) = \ln(x+1) + c \\ &\implies y+1 = K(x+1); \quad \text{with } K \in \mathbb{R}_+. \\ &\implies y = K(x+1) - 1; \quad \text{with } K \in \mathbb{R}_+. \end{aligned}$$

**Case  $n \neq 1$**  In this case we have

$$\begin{aligned} \int \frac{1}{y+1} dy = \int \frac{1}{(x+1)^n} dx &\implies \ln(y+1) = \frac{(x+1)^{1-n}}{1-n} + c \\ &\implies y+1 = K e^{f(x)}; \quad \text{with } f(x) = \frac{(x+1)^{1-n}}{1-n} \text{ and } K \in \mathbb{R}_+. \\ &\implies y = K e^{f(x)} - 1; \quad \text{with } f(x) = \frac{(x+1)^{1-n}}{1-n} \text{ and } K \in \mathbb{R}_+. \end{aligned}$$

While if we consider that the differential equation (2) is a linear differential equation with second member, so to solve it we must find its homogenous solution then its particular solution.

1. Solve the equation (2) according to the values of  $n$ .

(a) Let's find the homogenous solution of the given equation.

$$y' - \frac{y}{(x+1)^n} = 0 \implies \frac{y'}{y} = \frac{1}{(x+1)^n} \implies \int \frac{1}{y} dy = \int \frac{1}{(x+1)^n} dx. \quad (3)$$

At this level we distinguish two possible cases, namely the Case  $n = 1$ : and the case  $n \neq 1$ .

**Case  $n = 1$ :** In the case (3) becomes

$$\begin{aligned}\int \frac{1}{y} dy &= \int \frac{1}{(x+1)} dx \implies \ln(y) = \ln(x+1) + c \\ &\implies y_h = K(x+1); \text{ with } K \in \mathbb{R}_+.\end{aligned}$$

**Case  $n \neq 1$ :** In this case we have

$$\begin{aligned}\int \frac{1}{y} dy &= \int \frac{1}{(x+1)^n} dx \implies \ln(y) = \frac{(x+1)^{1-n}}{1-n} + c \\ &\implies y_h = K e^{f(x)}; \text{ with } f(x) = \frac{(x+1)^{1-n}}{1-n} \text{ and } K \in \mathbb{R}_+.\end{aligned}$$

(b) Now let's find the particular solution of the equation by the variation of the constant.

**Case  $n = 1$ :** From the homogenous solution we deduce that the particular solution has the following form  $y_p = K(x)(x+1)$  this  $y'_p = K'(x)(x+1) + K(x)$ . So,

$$\begin{aligned}K'(x)(x+1) + K(x) - \frac{K(x)(x+1)}{(x+1)} &= \frac{1}{x+1} \implies K'(x) = \frac{1}{(1+x)^2} \\ &\implies K(x) = \frac{-1}{x+1} + c \text{ with } c \in \mathbb{R}. \\ &\implies y_g = y_h + y_p = c(x+1) - 1; \text{ with } c \in \mathbb{R}.\end{aligned}$$

**Case  $n \neq 1$ :** From the homogenous solution we deduce that the particular solution has the following form  $y_p = K(x)e^{f(x)}$  then  $y'_p = K'(x)e^{f(x)} + K(x)f'(x)e^{f(x)}$ , with  $f(x) = \frac{(x+1)^{1-n}}{1-n}$  and  $f'(x) = \frac{1}{(x+1)^n}$ . So,

$$\begin{aligned}K'(x)e^{f(x)} + K(x)f'(x)e^{f(x)} - K(x)e^{f(x)}f'(x) &= f'(x) \implies K'(x)e^{f(x)} = f'(x) \\ &\implies K'(x) = f'(x)e^{-f(x)} \\ &\implies K(x) = -e^{-f(x)} + c \\ &\implies y_g = ce^{f(x)} - 1.\end{aligned}$$

2. Determine the solution of (2) that satisfies the condition  $y(0) = 0$  when  $n = 1$ .

**Case  $n = 1$ :**

$$y(0) = 0 \implies c - 1 = 0 \implies c = 1 \implies y_g = x.$$

**Case  $n = 2$ :** When  $n = 2$ , we get  $y_g = ce^{\frac{-1}{x+1}} - 1$  and as  $y(0) = 0$  then  $c = e^1 \implies y_g = e^{\frac{x}{x+1}} - 1$ .

## Solution of the Exercise 4

Let's solve the following equation:

$$y'' + 2y' - 3y = 3x + 4 - 4e^{2x} + 5\sin(2x + 1). \quad (4)$$

**Homogenous solution:** The homogenous solution is the solution of the following equation:

$$y'' + 2y' - 3y = 0 \implies R^2 + 2R - 3 = 0 \implies \Delta = 16 > 0 \implies R_1 = 1 \text{ and } R_2 = -3.$$

As the equation admits a double solution then the homogenous solution is given by follow:

$$y_h = C_1 e^x + C_2 e^{-3x}.$$

**Particulars solutions** From the given equation we can distinguish three particular solutions.

1. The first particular solution corresponding to  $3x + 4$  have the form  $y_{p_1} = ax + b$  so,  $y'_{p_1} = a$  and  $y''_{p_1} = 0$ .  
By substitution of  $y_{p_1}$ ,  $y'_{p_1}$  and  $y''_{p_1}$  in (4) we get

$$0 + 2a - 3(ax + b) = 3x + 4 \implies \begin{cases} -3a & = 3 \\ 2a - 3b & = 4 \end{cases} \implies \begin{cases} a & = -1 \\ b & = -2 \end{cases}$$

$$\implies y_{p_1} = -x - 2.$$

2. The second particular solution corresponding to  $-4e^{-3x}$  have the form  $y_{p_2} = axe^{-3x}$  so,  $y'_{p_2} = (-3ax + a)e^{-3x}$  and  $y''_{p_2} = (9ax - 6a)e^{-3x}$ . By substitution of  $y_{p_2}$ ,  $y'_{p_2}$  and  $y''_{p_2}$  in (4) we get

$$((9ax - 6a)e^{-3x}) + 2((-3ax + a)e^{-3x}) - 3(axe^{-3x}) = -4e^{-3x} \implies -4a = -4 \implies a = 1$$

$$\implies y_{p_2} = xe^{-3x}.$$

3. The third particular solution corresponding to  $\sin(2x + 1)$  have the form  $y_{p_3} = a \sin(2x + 1) + b \cos(2x + 1)$  so,  $y'_{p_3} = -2b \sin(2x + 1) + 2a \cos(2x + 1)$  and  $y''_{p_3} = -4a \sin(2x + 1) - 4b \cos(2x + 1)$ .

By substitution of  $y_{p_3}$ ,  $y'_{p_3}$  and  $y''_{p_3}$  in (4) we get

$$(-7a - 4b) \sin(2x + 1) + (4a - 7b) \cos(2x + 1) = 5 \sin(2x + 1)$$

$$\implies \begin{cases} 7a + 4b & = -5 \\ 4a - 7b & = 0 \end{cases}$$

$$\implies \begin{cases} a & = \frac{-7}{13} \\ b & = \frac{-4}{13} \end{cases}$$

$$\implies y_{p_3} = -\frac{7}{13} \sin(2x + 1) - \frac{4}{13} \cos(2x + 1)$$

We conclude that the global solution of the equation is

$$y_g = y_h + y_{p_1} + y_{p_2} + y_{p_3}$$

$$= C_1 e^x + (C_2 + x)e^{-3x} - x - 2 - \frac{7}{13} \sin(2x + 1) - \frac{4}{13} \cos(2x + 1).$$