Chapter 7

Matrices

In 1855, Arthur Cayley introduced the matrix as a representation of linear elements, and this period is considered the beginning of linear algebra and matrix theory. Matrices and their applications are used in most scientific fields, in every branch of physics, such as mechanics, engineering optics, electromagnetism, quantum mechanics, and for studying physical phenomena such as the movement of solid bodies, as well as in computer graphics, processing three-dimensional models and displaying them on a two-dimensional screen, as well as in probability theories and statistics, and in economics, they are used to describe systems of economic relations.

7.1 *Definitions*

Definition 7.1

Let n and p be two non-zero natural numbers.

(1 The matrix A is a rectangular table of elements of the field $\mathbb{K}$ which can be the set of real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$.

(2 A is of order or of class $n \times p$ if the table consists of n rows and p columns.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix} \quad \text{or} \quad A = (a_{ij})_{1\le i\le n; 1\le j\le p}\,.$$

(3 The elements of the table are called the coefficients of the matrix A.

(4 The coefficient at the intersection of row i and column j is denoted by a_{ij}.

Example 7.1

$$A = \begin{pmatrix} 5 & -2 \\ 0 & 3 \\ 1 & 9 \end{pmatrix}$$

It is a matrix of class 3×2 i.e., three rows and two columns. For example, $a_{11} = 5$ and $a_{22} = 3$.

Example 7.2

(1 The matrix

$$A = \begin{pmatrix} 1 & 17 & 0 \\ \frac{1}{2} & \sqrt{5} & 5 \end{pmatrix}$$

is a 2×3 matrix consisting of two rows and three columns.

(2 a_{23} is the coefficient at the intersection of the second row and the third column. It is equal to 5.

Definition 7.2

The set of matrices containing n rows and p columns with coefficients in $\mathbb{K}$ is denoted by $M_{n,p}(\mathbb{K})$. The vector space elements of $M_{n,p}(\mathbb{R})$ are called real matrices.

7.1.1 Special matrices

Here are some interesting types of matrices:

(1 If $n = p$ (the number of rows is equal to the number of columns), in this case we say that the matrix is square. We then denote the set of matrices by $M_n(\mathbb{K})$ instead of $M_{n,n}(\mathbb{K})$.

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

The elements $a_{11}, a_{22}, \dots, a_{nn}$ make up the diagonal of the matrix.

(2 If $p = 1$, then A is a column matrix:

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

(3 If $n = 1$, then A is a row matrix:

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \end{pmatrix}.$$

(4 A matrix (of class $n \times p$) whose coefficients are all zeros is called a zero matrix, or null matrix, and is denoted by $0_{n,p}$ or more simply 0. In matrix arithmetic, the zero matrix plays the role of the number 0 for real numbers.

Example 7.3

(1 The matrix

$$M = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

is a column matrix.

(2 The matrix

$$N = \begin{pmatrix} -1 & 5 & 3 & 5 \end{pmatrix}$$

is a row matrix.

(3 The matrix

$$P = \begin{pmatrix} 2 & 4 & -3 \\ 0 & -1 & 6 \\ -4 & 0 & \pi \end{pmatrix}$$

is a square matrix of order 3.

(4 The matrix

$$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is the zero matrix or the null matrix.

The following square matrix is called the identity matrix

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Its diagonal elements are 1 and all its other elements are 0. We denote it by I_n or simply by I.

In matrix arithmetic, the identity matrix plays a role similar to that of the number 1 for real numbers. It is the neutral element for multiplication.

Proposition 7.1.1. *If A is a matrix of class $n \times p$ then*

$$I_n \cdot A = A \quad \text{and} \quad A \cdot I_p = A.$$

(5 A square matrix A is symmetric if it is equal to its transpose, that is, if we have:

$$A = A^T,$$

or if $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots, n$. In other words, the coefficients of the matrix are symmetric with respect to the diagonal.

Example 7.4

The following matrices are symmetric matrices:

$$\begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 5 \\ 0 & 2 & -1 \\ 5 & -1 & 0 \end{pmatrix}$$

(6 A square matrix A is antisymmetric if we have:

$$A^T = -A,$$

or if $a_{ij} = -a_{ji}$ for all $i, j = 1, \ldots, n$.

Example 7.5

$$\begin{pmatrix} 0 & -5 \\ 5 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 4 & 9 \\ -4 & 1 & -3 \\ -9 & 3 & 2 \end{pmatrix}$$

7.1.2 Equal matrices

Let n and p be non-zero natural numbers, and let the matrices A and B be of the same class $n \times p$.

Definition 7.3

We say that the two matrices A and B are equal if their corresponding elements are equal, and we write:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix} = B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix} \Longleftrightarrow \forall i, j : a_{ij} = b_{ij}$$

Example 7.6

Let the two matrices be A and B where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}, \quad B = \begin{pmatrix} 2 & \sqrt{3} & \pi \\ 2i & 7 & \frac{1}{2} \end{pmatrix}$$

We say that A is equal to B if

$$\begin{aligned} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} &= \begin{pmatrix} 2 & \sqrt{3} & \pi \\ 2i & 7 & \frac{1}{2} \end{pmatrix} \\ &\Leftrightarrow \begin{cases} a_{11} = 2, \quad a_{12} = \sqrt{3}, \quad a_{13} = \pi, \\ a_{21} = 2i, \quad a_{22} = 7, \quad a_{23} = \frac{1}{2}. \end{cases} \end{aligned}$$

Proposition 7.1.2. *The two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of class n×p are equal if and only if $a_{ij} = b_{ij}$ for each i, j.*

7.2 *Matrix Calculus*

7.2.1 Product of a matrix by a scalar

Proposition 7.2.1. *If we have the matrix $A = (a_{ij})$ and a scalar $\lambda \in \mathbb{R}$, we define λA as the matrix $C = (c_{ij})$ where $c_{ij} = \lambda a_{ij}$ for each i, j.*

Example 7.7

Let the matrix

$$A = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & -\frac{3}{4} \end{pmatrix},$$

then

$$-2A = \begin{pmatrix} -2 \cdot \frac{1}{2} & -2 \cdot 1 \\ -2 \cdot 0 & -2 \cdot (-\frac{3}{4}) \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 0 & \frac{3}{2} \end{pmatrix}.$$

7.2.2 Matrix addition

Proposition 7.2.2. *If we have $A = (a_{ij})$ and $B = (b_{ij})$, two matrices of class n×p, we define the sum of the two matrices denoted by $A + B$ as the matrix $C = (c_{ij})$ of class n×p where $c_{ij} = a_{ij} + b_{ij}$ for each i, j.*

Example 7.8

The sum of two matrices of class 2×3:

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 4 \end{pmatrix} + \begin{pmatrix} 0 & -1 & -2 \\ -3 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 1+0 & 0-1 & -1-2 \\ 2-3 & 1-1 & 4+5 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -3 \\ -1 & 2 & 9 \end{pmatrix}$$

Proposition 7.2.3. *Let A, B and C be three matrices of the set $M_{n,p}(\mathbb{K})$, and let $\alpha \in \mathbb{K}$ and $\beta \in \mathbb{K}$ be scalars.*

(1 *Addition is commutative:*

$$A + B = B + A,$$

(2 *Addition is associative:*

$$A + (B + C) = (A + B) + C,$$

(3 *The null matrix is the neutral element with respect to addition in the set of matrices:*

$$A + 0 = A,$$

(4 $(\alpha + \beta)A = \alpha A + \beta A$,

(5 $\alpha(A + B) = \alpha A + \alpha B$.

Example 7.9

Let

$$A = \begin{pmatrix} 3 & -2 \\ 1 & 7 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 5 \\ 2 & -1 \end{pmatrix} \quad \text{then} \quad A + B = \begin{pmatrix} 3 & 3 \\ 3 & 6 \end{pmatrix}.$$

But, if:

$$C = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$$

then $A + C$ is undefined.

7.2.3 Matrix multiplication

Proposition 7.2.4. *Let $A = (a_{ij})$ be of class $n \times p$ and $B = (b_{jk})$ be of class $p \times q$. We define the product AB (also denoted by AB) as the matrix $C = (c_{ik})$ defined as follows:*

$$c_{ik} = \sum_{j=1}^{p} a_{ij} b_{jk}, \quad \forall i, k : \ 1 \le i \le n \quad \textit{and} \quad 1 \le k \le q.$$

We can write the coefficient in a more detailed way:

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ik} b_{kj} + \cdots + a_{ip} b_{pj}.$$

Remark 7.2.1. *A product is defined only if the number of columns in A equals the number of rows in B. This is why matrix multiplication is generally not commutative.*

Example 7.10

Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

First, we verify the product can be performed: the class of the obtained matrix will be 2×2. Then we calculate each of the coefficients, starting with the first one:

$$c_{11} = 1 \cdot 1 \ + \ 2 \cdot (-1) \ + \ 3 \cdot 1 = 2,$$

then the rest:

$$AB = \begin{pmatrix} 2 & 7 \\ 3 & 11 \end{pmatrix}.$$

Example 7.11

We have:

$$c_{12} = 1 \cdot 1 + 2 \cdot 2 - 1 \cdot 3 = 2.$$

In the same way with the rest of the matrix elements, we get:

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 \\ 2 & 2 & 0 \\ -1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 0 \\ -3 & 10 & 4 \end{pmatrix}$$

The product of two non-null matrices can be zero. In other words, we could have $A \neq 0$ and $B \neq 0$ but $AB = 0$.

Example 7.12

Let

$$A = \begin{pmatrix} 0 & -1 \\ 0 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -3 \\ 0 & 0 \end{pmatrix} \quad \text{so} \quad AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Remark 7.2.2. *$AB = AC$ does not imply $B = C$. It is possible to have $AB = AC$ and $B \neq C$.*

Example 7.13

Let

$$A = \begin{pmatrix} 0 & -1 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -1 \\ 5 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 5 \\ 5 & 4 \end{pmatrix} \quad \text{so} \quad AB = AC = \begin{pmatrix} -5 & -4 \\ 15 & 12 \end{pmatrix}.$$

Properties

Proposition 7.2.5. (1 *Multiplication is associative:*

$$A(BC) = (AB)C.$$

(2 *Multiplication is distributive over addition:*

$$A(B + C) = AB + AC \quad \textit{and} \quad (B + C)A = BA + CA$$

(3

$$A \cdot 0 = 0 \quad \textit{and} \quad 0 \cdot A = 0.$$

7.2.4 Transpose of a matrix

Definition 7.4

The transpose of a matrix is a matrix derived from a given matrix by turning its rows into columns and its columns into rows, i.e., by swapping rows and columns in the following way:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} \longrightarrow \begin{pmatrix} a_{11} & b_{21} \\ a_{12} & b_{22} \\ a_{13} & b_{23} \end{pmatrix},$$

We denote the transpose of a matrix A by A^T.

Remark 7.2.3. *The transpose of a matrix of class* $n \times p$ *produces a new matrix of class* $p \times n$.

Example 7.14

We have

$$\begin{pmatrix} 4 & 2 & 3 \\ 4 & 5 & -6 \\ -7 & 8 & 0 \end{pmatrix}^T = \begin{pmatrix} 4 & 4 & -7 \\ 2 & 5 & 8 \\ 3 & -6 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3 \\ 1 & -5 \\ -1 & 2 \end{pmatrix}^T = \begin{pmatrix} 0 & 1 & -1 \\ 3 & -5 & 2 \end{pmatrix}, \qquad (3 \quad -2 \quad 5)^T = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$$

Properties

Theorem 7.1

- $(A + B)^T = A^T + B^T$.
- $(\alpha A)^T = \alpha A^T$.

- $(A^T)^T = A$.
- $(AB)^T = B^T A^T$.
- If A is invertible, then A^T is also invertible and we have:

$$(A^T)^{-1} = (A^{-1})^T.$$

7.3 *Square Matrices*

The matrices we will study in the following are square matrices of $\mathscr{M}_n(\mathbb{K})$.

7.3.1 Matrix trace

In the case of a square matrix of type $n \times n$, the elements $(a_{11}, a_{22}, \ldots, a_{nn})$ are called diagonal elements.

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

Definition 7.5

The trace of the matrix A is the sum of the diagonal elements of the matrix A. In other words:

$$\boxed{\mathbf{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}.}$$

Example 7.15

- If we have

  $$A = \begin{pmatrix} 2 & 7 \\ 0 & 5 \end{pmatrix},$$

  then

  $$\mathbf{tr}(A) = 2 + 5 = 7.$$

- For

  $$B = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 8 \\ 11 & 0 & -10 \end{pmatrix}, \text{ then } \mathbf{tr}(B) = 1 + 2 - 10 = -7.$$

Properties

Theorem 7.2

Let A and B be matrices of class $n \times n$. Then:

- $\mathbf{tr}(A+B) = \mathbf{tr}(A) + \mathbf{tr}(B)$,
- $\mathbf{tr}(A^T) = \mathbf{tr}(A)$,
- $\mathbf{tr}(AB) = \mathbf{tr}(BA)$,
- For every $\alpha \in \mathbb{K}$,
$$\mathbf{tr}(\alpha A) = \alpha \mathbf{tr}(A)$$

7.3.2 Determinant of a square matrix

The determinant of a square matrix is a numerical value that gives us concise information about the matrix, such as whether it is invertible. It is useful to know this information before attempting to perform any algebraic operation involving the matrix. We always prefer to minimize the number of computational steps needed to achieve this.

In what follows, we consider matrices with coefficients in a commutative field $\mathbb{K}$, which can be $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We will explain how to calculate the determinant of a matrix with small dimensions.

Definition 7.6

Let A be a square matrix of $\mathscr{M}_n(\mathbb{K})$ where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

We call the determinant of the matrix A the number in $\mathbb{K}$ which we denote by $\det(A)$ or $|A|$ and write:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

We will demonstrate several key properties of determinants. Each time we will give simple explanations for each new concept. When calculating the determinant of a 4×4 or higher order matrix, a significant number of calculations are required. So we will mostly use 2×2 or 3×3 matrices. Before we get started, we'll briefly review how to compute the determinants of these two types of matrices, starting with 2 × 2 matrices.

Determinants of dimension 2 and 3

In dimension 2, it is very easy to compute the determinant. The value of the determinant of a 2×2 matrix is computed as follows:

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

More clearly, the process is as follows:

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

Unlike the 2×2 case, when calculating the determinant of a higher-order matrix, there is more than one option to proceed with the calculation.

Let $A \in \mathscr{M}_3(\mathbb{K})$ be a 3×3 matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

The formula for the determinant can be expressed by expansion:

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

There is another convenient method, known as Sarrus's rule, which only works for 3×3 matrices: We copy the first two columns to the right of the matrix, then add the products of the three diagonals from top-left to bottom-right, and subtract the products of the three diagonals from bottom-left to top-right.

Example 7.16

Let's calculate the determinant of the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

.

According to Sarrus's rule:

$$\begin{aligned} \det A &= 2 \cdot (-1) \cdot 1 + 1 \cdot 3 \cdot 3 + 0 \cdot 1 \cdot 2 \\ &\quad - (3 \cdot (-1) \cdot 0 + 2 \cdot 3 \cdot 2 + 1 \cdot 1 \cdot 1) \\ &= (-2) + 9 + 0 - (0 + 12 + 1) = 7 - 13 = -6. \end{aligned}$$

Properties

Theorem 7.3

Let A and B be two matrices of class $n \times n$. Then:

- $\det(A^T) = \det(A)$,
- $\det(AB) = \det(A)\det(B)$,
- $\det(A^{-1}) = 1/\det(A)$ (if A is invertible),
- For every $\alpha \in \mathbb{K}$,
$$\det(\alpha A) = \alpha^n \det(A)$$
(Note: Correction from the original text which incorrectly stated $\alpha \det(A)$).

Note: The property $\det(A + B) = \det(A) + \det(B)$ is generally false.

7.3.3 Similar matrices

Definition 7.7

Let A and B be matrices of the set $\mathscr{M}_n(\mathbb{K})$. We say that the matrix B is similar to the matrix A if there exists an invertible matrix $P \in \mathscr{M}_n(\mathbb{K})$ such that:

$$B = P^{-1}AP.$$

We can easily prove that the following relation is an equivalence relation on the set $\mathscr{M}_n(\mathbb{K})$.

$$\forall A, B \in M_n(\mathbb{K}) : \quad A\mathscr{R}B \Longleftrightarrow A \text{ is Similar to } B$$

- Reflexive: the matrix A is similar to itself.
- Symmetric: if A is similar to the matrix B then B is similar to the matrix A.
- Transitive: if A is similar to the matrix B and B is similar to the matrix C then A is similar to the matrix C.

We then say that the two matrices A and B are similar.

Remark 7.3.1. *Two matrices are similar if they represent the same endomorphism, but are expressed in different bases.*

7.3.4 Matrix inverse

Definition 7.8

Let A be a square matrix of degree n. If there exists a square matrix B of degree n such that:

$$AB = I \quad \text{and} \quad BA = I,$$

we say that A is invertible. We call B the inverse of the matrix A and denote it by A^{-1}.

Remark 7.3.2. *In fact, it is sufficient to check only one of the following conditions:* $AB = I$ *or* $BA = I$.

- In general, if A is invertible, for every $p \in \mathbb{N}$ we note:

$$A^{-p} = (A^{-1})^p = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{p \text{ times}}.$$

- The set of invertible matrices in $\mathscr{M}_n(\mathbb{K})$ is denoted by $GL_n(\mathbb{K})$.

Inverse of a matrix by the comparison method

Example 7.17

Let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

We study the invertibility of the matrix A, i.e., the existence of a matrix

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with coefficients in $\mathbb{K}$ such that $AB = I$ and $BA = I$. The condition $AB = I$ is equivalent to:

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \iff \begin{pmatrix} a+2c & b+2d \\ 3c & 3d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This equality is equivalent to the system:

$$\begin{cases} a + 2c = 1 \\ b + 2d = 0 \\ 3c = 0 \\ 3d = 1 \end{cases}$$

Its solution is: $a = 1$, $b = -\frac{2}{3}$, $c = 0$, $d = \frac{1}{3}$. Then, the matrix

$$B = \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}.$$

To prove it is suitable, it is also necessary to check the equality $BA = I$. The matrix A is invertible and its inverse is

$$A^{-1} = \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}.$$

Inverse of a matrix by the Gauss method

This method consists of performing several elementary row operations on the matrix A until it is transformed into the identity matrix I. The same elementary operations are performed simultaneously starting with the matrix I. We then end up with a matrix A^{-1}. We will explain this with easy-to-understand examples.

Practically, we do both operations at the same time by adopting the following layout: next to the matrix A that we want to invert, we place the identity matrix to form the augmented matrix $(A \mid I)$.

By performing elementary row operations on this augmented matrix until the left side becomes I, the right side becomes A^{-1}, i.e., we obtain the table $(I \mid B)$. Thus, $B = A^{-1}$.

We will use the following row operations:

(1 We can multiply any row by a non-zero scalar (any element of $\mathbb{K} \setminus \{0\}$).

$$L_i \leftarrow \lambda L_i, \quad \lambda \neq 0$$

(2 We can add to row L_i a multiple of another row L_j.

$$L_i \leftarrow L_i + \lambda L_j, \quad \lambda \in \mathbb{K}, \ (j \neq i)$$

(3 We can swap two rows.

$$L_i \leftrightarrow L_j.$$

Remark 7.3.3. *Remember that whatever you do on the left side of the augmented matrix, you must also do on the right side.*

Example 7.18

We calculate the inverse of the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 4 & 0 & -1 \\ -1 & 2 & 2 \end{pmatrix}.$$

The augmented matrix with numbered rows:

$$(A \mid I) = \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 4 & 0 & -1 & 0 & 1 & 0 \\ -1 & 2 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} L_1 \\ L_2 \\ L_3 \end{array}$$

We apply the Gauss method to create zeros in the first column. First, in the second row using the operation $L_2 \leftarrow L_2 - 4L_1$:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -8 & -5 & -4 & 1 & 0 \\ -1 & 2 & 2 & 0 & 0 & 1 \end{array}\right)$$

Then, create a zero in the first column of the third row with $L_3 \leftarrow L_3 + L_1$:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -8 & -5 & -4 & 1 & 0 \\ 0 & 4 & 3 & 1 & 0 & 1 \end{array}\right)$$

Multiply row L_2 by $-1/8$ to obtain a leading 1:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{8} & \frac{1}{2} & -\frac{1}{8} & 0 \\ 0 & 4 & 3 & 1 & 0 & 1 \end{array}\right)$$

Continue the process to create zeros below the diagonal. Eliminate the element in the third row, second column with $L_3 \leftarrow L_3 - 4L_2$:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{8} & \frac{1}{2} & -\frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 1 \end{array}\right)$$

Then, scale the third row to get a leading 1 with $L_3 \leftarrow 2L_3$:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{8} & \frac{1}{2} & -\frac{1}{8} & 0 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right)$$

Now, perform the back substitution steps to create zeros above the diagonal. Eliminate above the diagonal in the second row with $L_2 \leftarrow L_2 - \frac{5}{8}L_3$:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{7}{4} & -\frac{3}{4} & -\frac{5}{4} \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right)$$

Finally, eliminate above the diagonal in the first row with $L_1 \leftarrow L_1 - 2L_2 - L_3$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{7}{4} & -\frac{3}{4} & -\frac{5}{4} \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right)$$

Thus, the inverse of matrix A is the matrix obtained on the right. Factoring out $\frac{1}{4}$, we get:

$$A^{-1} = \frac{1}{4}\begin{pmatrix} -2 & 2 & 2 \\ 7 & -3 & -5 \\ -8 & 4 & 8 \end{pmatrix}$$

Finally, to verify the calculations, it is enough to check that $A \cdot A^{-1} = I$.

Inverse of a matrix by the adjugate matrix method

Definition 7.9

Let A be an $n \times n$ matrix: We create the submatrix A_{ij} by deleting the i-th row and the j-th column from A, resulting in an $(n-1) \times (n-1)$ matrix. We call the adjugate matrix of A, denoted by $\operatorname{adj}(A)$ (sometimes A^*), the transpose of the cofactor matrix. Its elements are given by:

$$(\operatorname{adj}(A))_{ij} = C_{ji} = (-1)^{i+j} \det(A_{ji}),$$

where C_{ji} is the cofactor. Explicitly:

$$\operatorname{adj}(A) = \begin{pmatrix} +\det(A_{11}) & -\det(A_{21}) & +\det(A_{31}) & \cdots \\ -\det(A_{12}) & +\det(A_{22}) & -\det(A_{32}) & \cdots \\ +\det(A_{13}) & -\det(A_{23}) & +\det(A_{33}) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Example 7.19

Let A be the matrix:

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

Then its adjugate matrix is:

$$\operatorname{adj}(A) = \begin{pmatrix} +\begin{vmatrix} 0 & 0 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} \\ -\begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} \\ +\begin{vmatrix} 3 & -1 \\ 0 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 0 & -2 & -2 \\ -2 & 2 & 4 \\ 0 & 2 & -6 \end{pmatrix}$$

Theorem 7.4

A square matrix A is invertible if and only if $\det(A) \neq 0$. If A is invertible, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

Example 7.20

Let matrix A be from the previous example:

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

Its adjugate matrix is:

$$\operatorname{adj}(A) = \begin{pmatrix} 0 & -2 & -2 \\ -2 & 2 & 4 \\ 0 & 2 & -6 \end{pmatrix}$$

Since the adjugate is already the transpose of the cofactor matrix, we can apply the theorem:

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{-4} \begin{pmatrix} 0 & -2 & -2 \\ -2 & 2 & 4 \\ 0 & 2 & -6 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -1 \\ \frac{1}{2} & -1 & \frac{3}{2} \end{pmatrix}.$$

7.4 Exercise Series N° 1

Exercise 7.1

Let

$$A = \begin{pmatrix} -7 & 2 \\ 0 & -1 \\ 1 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}, \quad D = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and

$$E = \begin{pmatrix} 1 & 2 \\ -3 & 0 \\ -8 & 6 \end{pmatrix}.$$

(A Calculate all possible sums of two of these matrices.

(B Calculate all possible products of two of these matrices.

(C Calculate $3A + 2E$ and $5B + 4EA^T$.

(D Find α such that $A - \alpha E$ is the null matrix.

Solution

(A The only possible sum of two of these matrices is

$$A + E = \begin{pmatrix} -6 & 4 \\ -3 & -1 \\ -7 & 2 \end{pmatrix}$$