Linear Algebra Notes

Chapter 2: Linear Maps

(Detailed theory, examples, and solved exercises)

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Contents

Chapter 1

Linear Maps

1.1 Definition and Basic Operations

1.1.1 Definition

Let V and W be vector spaces over the same field $\mathbb{F}$ (typically $\mathbb{R}$ or $\mathbb{C}$). A map (function) $T : V \to W$ is called **linear** (or a **linear map / linear transformation**) if, for all $u, v \in V$ and all $\alpha \in \mathbb{F}$,

$$T(u + v) = T(u) + T(v), \tag{1.1}$$

$$T(\alpha u) = \alpha T(u). \tag{1.2}$$

Equivalently, $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$ for all $\alpha, \beta \in \mathbb{F}$ and $u, v \in V$.

1.1.2 Immediate consequences

If $T : V \to W$ is linear, then:

- $T(0_V) = 0_W$.
- $T(-v) = -T(v)$ for all $v \in V$.
- $T(u - v) = T(u) - T(v)$ for all $u, v \in V$.

1.1.3 Operations on linear maps

Let $T, S : V \to W$ be linear and let $a \in \mathbb{F}$.

- **Sum:** $(T + S)(v) = T(v) + S(v)$ is linear.
- **Scalar multiple:** $(aT)(v) = a\,T(v)$ is linear.
- **Composition:** If $R : W \to U$ is linear, then $(R \circ T) : V \to U$ defined by $(R \circ T)(v) = R(T(v))$ is linear.
- **Identity map:** $\mathrm{Id}_V : V \to V$ defined by $\mathrm{Id}_V(v) = v$ is linear.

1.1.4 Matrix representation (finite-dimensional case)

Assume $\dim(V) = n$ and $\dim(W) = m$, and choose bases

$$\mathcal{B} = (v_1, \dots, v_n) \text{ of } V, \qquad \mathcal{C} = (w_1, \dots, w_m) \text{ of } W.$$

For each j, write

$$T(v_j) = a_{1j} w_1 + \cdots + a_{mj} w_m.$$

The matrix $[T]_{\mathcal{C} \leftarrow \mathcal{B}} = (a_{ij}) \in \mathbb{F}^{m \times n}$ is called the **matrix of T in the bases** $(\mathcal{B}, \mathcal{C})$. If $x \in V$ has coordinate vector $[x]_{\mathcal{B}} \in \mathbb{F}^n$, then

$$[T(x)]_{\mathcal{C}} = [T]_{\mathcal{C} \leftarrow \mathcal{B}} \, [x]_{\mathcal{B}}.$$

1.1.5 Example 1 (Checking linearity)

Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (2x - y,\ x + 3y)$. This is linear because each component is a linear combination of x and y.

Define $S : \mathbb{R}^2 \to \mathbb{R}$ by $S(x, y) = x^2 + y$. This is *not* linear since $S(2x, 2y) = 4x^2 + 2y \neq 2(x^2 + y) = 2S(x, y)$ in general.

1.1.6 Example 2 (Operations)

Let $T(x, y) = (x, y)$ and $S(x, y) = (y, x)$ on $\mathbb{R}^2$. Then $(T + S)(x, y) = (x + y,\ y + x)$ is linear, and $(S \circ S) = T$ (since swapping twice returns the original vector).

1.2 Kernel and Image

1.2.1 Definitions

Let $T : V \to W$ be linear.

- The **kernel** (or **null space**) of T is

$$\ker(T) = \{v \in V :\ T(v) = 0_W\}.$$

- The **image** (or **range**) of T is

$$\operatorname{Im}(T) = T(V) = \{T(v) :\ v \in V\} \subseteq W.$$

1.2.2 Subspace properties

Theorem 1.1. *For a linear map $T : V \to W$, $\ker(T)$ is a subspace of V and $\operatorname{Im}(T)$ is a subspace of W.*

Proof. **Kernel:** If $u, v \in \ker(T)$ then $T(u) = T(v) = 0$ and $T(u + v) = T(u) + T(v) = 0$, so $u + v \in \ker(T)$. Also, for any $\alpha \in \mathbb{F}$, $T(\alpha u) = \alpha T(u) = 0$, so $\alpha u \in \ker(T)$. Hence $\ker(T)$ is a subspace.

Image: If $y_1 = T(u)$ and $y_2 = T(v)$ are in $\operatorname{Im}(T)$, then $y_1 + y_2 = T(u) + T(v) = T(u+v) \in \operatorname{Im}(T)$, and $\alpha y_1 = \alpha T(u) = T(\alpha u) \in \operatorname{Im}(T)$. So $\operatorname{Im}(T)$ is a subspace of W. □

1.2.3 Computing kernel and image via matrices

If $T(x) = Ax$ for a matrix $A \in \mathbb{F}^{m\times n}$ (standard bases), then:

$$\ker(T) = \{x \in \mathbb{F}^n : \ Ax = 0\}, \qquad \operatorname{Im}(T) = \{Ax : \ x \in \mathbb{F}^n\} = \operatorname{Col}(A),$$

where $\operatorname{Col}(A)$ is the column space of A.

1.2.4 Example 3 (Kernel and image with full solution)

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$T(x, y, z) = \big(x + y + z,\ 2x + y + 3z\big).$$

In matrix form,

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \end{pmatrix}, \qquad T(\mathbf{x}) = A\mathbf{x}.$$

Kernel. Solve $A(x, y, z)^\mathsf{T} = 0$:

$$\begin{cases} x + y + z = 0, \\ 2x + y + 3z = 0. \end{cases}$$

Subtract the first equation from the second:

$$(2x + y + 3z) - (x + y + z) = x + 2z = 0 \ \Rightarrow\ x = -2z.$$

From $x + y + z = 0$:

$$-2z + y + z = 0 \Rightarrow y = z.$$

Let $z = t$. Then $(x, y, z) = (-2t,\ t,\ t) = t(-2, 1, 1)$. Hence

$$\ker(T) = \operatorname{span}\{(-2, 1, 1)\}.$$

Image. $\operatorname{Im}(T) = \operatorname{Col}(A) = \operatorname{span}\{(1, 2)^\mathsf{T},\ (1, 1)^\mathsf{T},\ (1, 3)^\mathsf{T}\}$. The first two columns are independent because

$$\det \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = 1 \cdot 1 - 2 \cdot 1 = -1 \neq 0.$$

Thus they form a basis of the image, so

$$\operatorname{Im}(T) = \mathbb{R}^2.$$

1.3 Rank of a Linear Map

1.3.1 Definition

If $T : V \to W$ is linear and V is finite-dimensional, the **rank** of T is

$$\operatorname{rank}(T) = \dim(\operatorname{Im}(T)).$$

The **nullity** of T is

$$\operatorname{nullity}(T) = \dim(\ker(T)).$$

1.3.2 Rank of a matrix

If $T(\mathbf{x}) = A\mathbf{x}$, then

$$\operatorname{rank}(T) = \operatorname{rank}(A) = \dim(\operatorname{Col}(A)),$$

equal to the number of pivot columns in the reduced row echelon form of A.

1.3.3 Example 4 (Rank and nullity)

Using Example 3:

$$\ker(T) = \operatorname{span}\{(-2, 1, 1)\} \Rightarrow \operatorname{nullity}(T) = 1, \qquad \operatorname{Im}(T) = \mathbb{R}^2 \Rightarrow \operatorname{rank}(T) = 2.$$

1.4 Rank Theorem (Rank–Nullity Theorem)

Theorem 1.2 (Rank–Nullity). *Let $T : V \to W$ be a linear map and assume $\dim(V) = n < \infty$. Then*

$$\dim(V) = \operatorname{rank}(T) + \operatorname{nullity}(T).$$

Proof. Let $(k_1, \ldots, k_r)$ be a basis of $\ker(T)$, so $r = \operatorname{nullity}(T)$. Extend it to a basis of V:

$$(k_1, \ldots, k_r,\ v_{r+1}, \ldots, v_n).$$

Consider the vectors $T(v_{r+1}), \ldots, T(v_n)$ in W. They span $\operatorname{Im}(T)$ and are linearly independent; hence they form a basis of $\operatorname{Im}(T)$. Therefore $\operatorname{rank}(T) = n - r$, so $n = \operatorname{rank}(T) + \operatorname{nullity}(T)$. □

1.4.1 Example 5 (Using rank–nullity quickly)

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be linear. If $\operatorname{rank}(T) = 2$, then

$$4 = \operatorname{rank}(T) + \operatorname{nullity}(T) = 2 + \operatorname{nullity}(T) \Rightarrow \operatorname{nullity}(T) = 2.$$

1.5 Injective, Surjective, and Bijective Maps

1.5.1 Definitions

Let $T : V \to W$ be a function.

- **Injective** (one-to-one): $T(u) = T(v) \Rightarrow u = v$.
- **Surjective** (onto): for every $w \in W$, there exists $v \in V$ such that $T(v) = w$.
- **Bijective**: both injective and surjective.

1.5.2 Characterization using the kernel and rank

Theorem 1.3. *Let $T : V \to W$ be linear.*

1. *T is injective* $\iff \ker(T) = \{0\} \iff \mathrm{nullity}(T) = 0$.
2. *If $\dim(W) = m < \infty$, then T is surjective* $\iff \mathrm{Im}(T) = W \iff \mathrm{rank}(T) = m$.
3. *If $\dim(V) = \dim(W) = n < \infty$, then the following are equivalent:*

$$T \text{ injective} \iff T \text{ surjective} \iff T \text{ bijective} \iff \mathrm{rank}(T) = n.$$

Proof. (1) $T(u) = T(v) \iff T(u - v) = 0 \iff u - v \in \ker(T)$. Thus injectivity is equivalent to $\ker(T) = \{0\}$.

(2) Surjectivity means $\mathrm{Im}(T) = W$. In finite dimension, this is equivalent to $\dim(\mathrm{Im}(T)) = \dim(W)$, i.e., $\mathrm{rank}(T) = m$.

(3) If $\dim(V) = \dim(W) = n$, then by rank–nullity:

$$n = \mathrm{rank}(T) + \mathrm{nullity}(T).$$

So $\mathrm{nullity}(T) = 0 \iff \mathrm{rank}(T) = n$. Using (1) and (2), injectivity $\iff$ surjectivity $\iff$ bijectivity. □

1.5.3 Example 6 (Injective or surjective?)

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the map from Example 3. It was found that $\ker(T) \neq \{0\}$, so T is **not injective**. Also $\mathrm{Im}(T) = \mathbb{R}^2$, so T is **surjective**.

1.5.4 Example 7 (Square matrix test)

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x) = Ax$ with

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

$\det(A) = 1 \cdot 1 \cdot 3 = 3 \neq 0$. Therefore A is invertible, $\mathrm{rank}(A) = 3$, and T is bijective.

1.6 Worked Examples (Solutions Included)

1.6.1 Example 8 (Kernel, image, rank, injectivity, surjectivity)

Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(x, y) = (x - 2y,\ 2x - 4y).$$

Matrix form:

$$A = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}.$$

Kernel. Solve $A(x,y)^\mathsf{T} = 0$:

$$\begin{cases} x - 2y = 0, \\ 2x - 4y = 0, \end{cases}$$

the second equation is redundant. Thus $x = 2y$. Let $y = t$, then $(x, y) = (2t, t) = t(2, 1)$. So

$$\ker(T) = \operatorname{span}\{(2,1)\}, \qquad \operatorname{nullity}(T) = 1.$$

Image and rank. Columns of A are $(1,2)^\mathsf{T}$ and $(-2,-4)^\mathsf{T} = -2(1,2)^\mathsf{T}$, so the column space is one-dimensional:

$$\operatorname{Im}(T) = \operatorname{span}\{(1,2)\}, \qquad \operatorname{rank}(T) = 1.$$

Injective/surjective. $\ker(T) \neq \{0\} \Rightarrow$ not injective. Also $\operatorname{rank}(T) = 1 < 2 = \dim(\mathbb{R}^2) \Rightarrow$ not surjective.

1.6.2 Example 9 (Linear map on polynomials)

Let $V = \mathcal{P}_3$ be the space of real polynomials of degree ≤ 3. Define $D : \mathcal{P}_3 \to \mathcal{P}_2$ by $D(p) = p'$ (derivative).

Linearity. $D(p+q) = (p+q)' = p'+q' = D(p)+D(q)$ and $D(\alpha p) = (\alpha p)' = \alpha p' = \alpha D(p)$.

Kernel. $D(p) = 0$ iff p is constant. Hence

$$\ker(D) = \{c : \ c \in \mathbb{R}\} \cong \mathbb{R}, \quad \operatorname{nullity}(D) = 1.$$

Image and rank. Every polynomial in $\mathcal{P}_2$ is the derivative of some polynomial in $\mathcal{P}_3$ (e.g., integrate). Hence $\operatorname{Im}(D) = \mathcal{P}_2$ and

$$\operatorname{rank}(D) = \dim(\mathcal{P}_2) = 3.$$

Check rank–nullity: $\dim(\mathcal{P}_3) = 4 = 3 + 1$.

Injective/surjective. Not injective, surjective onto $\mathcal{P}_2$.

1.6.3 Example 10 (Composition and kernels)

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be $T(x, y) = (x, 0)$ and $S : \mathbb{R}^2 \to \mathbb{R}$ be $S(u, v) = u$. Then $(S \circ T)(x, y) = S(x, 0) = x$.

Kernels.

$$\ker(T) = \{(0, y) : y \in \mathbb{R}\}, \qquad \ker(S) = \{(0, v) : v \in \mathbb{R}\}, \qquad \ker(S \circ T) = \{(0, y) : y \in \mathbb{R}\}.$$

1.7 Exercises (with solutions)

Exercise 1

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (x + y,\ y + z,\ x + z)$. Find $\ker(T)$ and decide whether T is injective.

Solution

Solve $T(x, y, z) = (0, 0, 0)$:

$$\begin{cases} x + y = 0, \\ y + z = 0, \\ x + z = 0. \end{cases}$$

From $x = -y$ and $z = -y$. Then $x + z = -y - y = -2y = 0 \Rightarrow y = 0$. Hence $x = 0$, $z = 0$ and

$$\ker(T) = \{(0, 0, 0)\}.$$

Therefore T is injective.

Exercise 2

Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \end{pmatrix}$. Consider $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x) = Ax$. Compute $\operatorname{rank}(T)$ and $\operatorname{nullity}(T)$.

Solution

Columns are $c_1 = (1, 2)^\mathsf{T}$, $c_2 = (0, 0)^\mathsf{T}$, $c_3 = (2, 4)^\mathsf{T} = 2c_1$. So the column space is spanned by c_1 and has dimension 1:

$$\operatorname{rank}(T) = 1.$$

By rank–nullity, $\dim(\mathbb{R}^3) = 3$ gives

$$3 = \operatorname{rank}(T) + \operatorname{nullity}(T) = 1 + \operatorname{nullity}(T) \Rightarrow \operatorname{nullity}(T) = 2.$$

Exercise 3

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be linear with $\operatorname{rank}(T) = 2$. Show that T is bijective.

Solution

Since $\dim(\mathbb{R}^2) = 2$ and $\operatorname{rank}(T) = 2 = \dim(\mathbb{R}^2)$, T is surjective. By rank–nullity,

$$2 = \operatorname{rank}(T) + \operatorname{nullity}(T) = 2 + \operatorname{nullity}(T) \Rightarrow \operatorname{nullity}(T) = 0,$$

so $\ker(T) = \{0\}$ and T is injective. Therefore T is bijective.