Detailed Solutions: Linear Maps (TD 02)

Solution to Exercise 1

Answer

Objective: Determine linearity by checking additivity $T(u+v) = T(u) + T(v)$ and homogeneity $T(\lambda u) = \lambda T(u)$.

a) **$T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T_1(x, y, z) = (x + y,\ y - z)$.** Let $u = (x, y, z)$ and $v = (x', y', z')$.

$$\begin{aligned} T_1(u+v) &= T_1(x+x', y+y', z+z') \\ &= ((x+x') + (y+y'), (y+y') - (z+z')) \\ &= (x+y+x'+y', y-z+y'-z') \\ &= (x+y, y-z) + (x'+y', y'-z') \\ &= T_1(u) + T_1(v). \end{aligned}$$

Also, $T_1(\lambda u) = (\lambda x + \lambda y, \lambda y - \lambda z) = \lambda(x+y, y-z) = \lambda T_1(u)$. **Conclusion:** T_1 is a **Linear Map**.

b) **$T_2 : \mathbb{R}^2 \to \mathbb{R}$ defined by $T_2(x, y) = xy$.** Check homogeneity with $\lambda = 2$ and $u = (1, 1)$:

$$T_2(2u) = T_2(2, 2) = 2 \cdot 2 = 4$$

$$2T_2(u) = 2(1 \cdot 1) = 2$$

Since $4 \neq 2$, $T_2(\lambda u) \neq \lambda T_2(u)$. **Conclusion:** T_2 is **Not Linear**.

c) **$T_3 : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T_3(x, y, z) = (x + 1,\ y,\ z)$.** Check the image of the zero vector. For linear maps, $T(0_{\mathbb{R}^3}) = 0_{\mathbb{R}^3}$.

$$T_3(0, 0, 0) = (0 + 1, 0, 0) = (1, 0, 0) \neq (0, 0, 0).$$

Conclusion: T_3 is **Not Linear**.

Solution to Exercise 2

Answer

Given $f(x, y, z) = (x + y + z, \quad 2x - z, \quad 3x + y)$.

1. **Kernel and Injectivity:** Solve $f(x, y, z) = (0, 0, 0)$.

$$\begin{cases} x + y + z = 0 & (1) \\ 2x - z = 0 \implies z = 2x & (2) \\ 3x + y = 0 \implies y = -3x & (3) \end{cases}$$

Substitute (2) and (3) into (1): $x + (-3x) + (2x) = 0 \implies 0 = 0$. This is true for any x. The solutions are of the form $(x, -3x, 2x) = x(1, -3, 2)$.

$$\text{Ker}(f) = \text{span}\{(1, -3, 2)\}$$

Since $\text{Ker}(f) \neq \{0_{\mathbb{R}^3}\}$, f **is not injective**.

2. **Image and Surjectivity:** We calculate images of the canonical basis vectors:
   - $f(1, 0, 0) = (1, 2, 3)$
   - $f(0, 1, 0) = (1, 0, 1)$
   - $f(0, 0, 1) = (1, -1, 0)$

   Notice that $(1, -1, 0) = 2(1, 0, 1) - (1, 2, 3)$ is false... wait, check dependence. From kernel, $1 \cdot e_1 - 3 \cdot e_2 + 2 \cdot e_3 \in \text{Ker}(f)$, so $1 \cdot f(e_1) - 3 \cdot f(e_2) + 2 \cdot f(e_3) = 0$. Thus the vectors are dependent. We only need two independent vectors. $(1, 2, 3)$ and $(1, 0, 1)$ are not proportional.

   $$\text{Im}(f) = \text{span}\{(1, 2, 3), (1, 0, 1)\}$$

   Since $\dim(\text{Im}f) = 2 < \dim(\mathbb{R}^3) = 3$, f **is not surjective**.

3. **Rank-Nullity Theorem:** Theorem: $\dim(E) = \dim(\text{Ker}f) + \dim(\text{Im}f)$.

   $$3 = 1 + 2$$

   The theorem is verified.

Solution to Exercise 3 (Linear Map on $\mathbb{R}^3$)

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by:

$$f(x, y, z) = (x - 2y, \quad y + z, \quad 0)$$

Answer

1. **Show that f is a linear map.**
   Let $u = (x, y, z)$ and $v = (x', y', z')$ be vectors in $\mathbb{R}^3$, and $\lambda \in \mathbb{R}$ be a scalar.
   - **Additivity:**

   $$\begin{aligned} f(u + v) &= f(x + x', y + y', z + z') \\ &= ((x + x') - 2(y + y'), \quad (y + y') + (z + z'), \quad 0) \\ &= ((x - 2y) + (x' - 2y'), \quad (y + z) + (y' + z'), \quad 0) \\ &= (x - 2y, y + z, 0) + (x' - 2y', y' + z', 0) \\ &= f(u) + f(v) \end{aligned}$$

   - **Homogeneity:**

   $$\begin{aligned} f(\lambda u) &= f(\lambda x, \lambda y, \lambda z) \\ &= (\lambda x - 2(\lambda y), \quad \lambda y + \lambda z, \quad 0) \\ &= (\lambda(x - 2y), \quad \lambda(y + z), \quad 0) \\ &= \lambda f(u) \end{aligned}$$

Since both conditions are satisfied, f **is a linear map.**

2. Determine $\text{Ker}(f)$ **and** $\text{Im}(f)$ **and their dimensions.**

1. **Kernel** ($\text{Ker}f$)**:** We solve $f(x,y,z) = (0,0,0)$:

$$\begin{cases} x - 2y = 0 \implies x = 2y \\ y + z = 0 \implies z = -y \\ 0 = 0 \end{cases}$$

The variable y is free. The solution space is:

$$\text{Ker}(f) = \{(2y, y, -y) \mid y \in \mathbb{R}\} = \text{span}\{(2,1,-1)\}$$

Dimension: $\dim(\text{Ker}f) = 1$.

2. **Image** ($\text{Im}f$)**:** The image is generated by the images of the standard basis vectors:

- $f(e_1) = f(1,0,0) = (1,0,0)$
- $f(e_2) = f(0,1,0) = (-2,1,0)$
- $f(e_3) = f(0,0,1) = (0,1,0)$

Notice that $(-2,1,0) = -2(1,0,0) + (0,1,0)$, so the vectors are linearly dependent. The vectors $(1,0,0)$ and $(0,1,0)$ are linearly independent.

$$\text{Im}(f) = \text{span}\{(1,0,0), (0,1,0)\}$$

Dimension: $\dim(\text{Im}f) = 2$.

3. Rank and Rank-Nullity Theorem.

- **Rank:** By definition, $\text{rank}(f) = \dim(\text{Im}f) = 2$.
- **Verification:** The Rank-Nullity Theorem states:

$$\dim(\mathbb{R}^3) = \dim(\text{Ker}f) + \text{rank}(f)$$

$$3 = 1 + 2$$

The theorem is verified.

4. Injectivity, Surjectivity, Bijectivity.

- **Injective?** No, because $\text{Ker}(f) \neq \{0_{\mathbb{R}^3}\}$ (it contains $(2,1,-1)$).
- **Surjective?** No, because $\dim(\text{Im}f) = 2 \neq \dim(\mathbb{R}^3) = 3$.
- **Bijective?** No, because it is neither injective nor surjective.

Solution to Exercise 4 (Polynomial Map)

Let $f : \mathbb{R}_2[X] \to \mathbb{R}_2[X]$ be defined by $f(P(X)) = P(X+1) - P(X)$. Let $\mathcal{B} = (1, X, X^2)$.

Answer

1. **Verify that f is a linear map.**
   Let $P, Q \in \mathbb{R}_2[X]$ and $\lambda \in \mathbb{R}$.

$$\begin{aligned} f(\lambda P + Q) &= (\lambda P + Q)(X+1) - (\lambda P + Q)(X) \\ &= [\lambda P(X+1) + Q(X+1)] - [\lambda P(X) + Q(X)] \\ &= \lambda[P(X+1) - P(X)] + [Q(X+1) - Q(X)] \\ &= \lambda f(P) + f(Q) \end{aligned}$$

Thus, f is linear.

2. **Determine the kernel of f.**
   We seek polynomials $P \in \mathbb{R}_2[X]$ such that $f(P) = 0$, i.e., $P(X+1) = P(X)$. Let $P(X) = aX^2 + bX + c$.

$$\begin{aligned} P(X+1) &= a(X+1)^2 + b(X+1) + c \\ &= a(X^2 + 2X + 1) + bX + b + c \\ &= aX^2 + (2a+b)X + (a+b+c) \end{aligned}$$

Setting $P(X+1) = P(X)$:

$$aX^2 + (2a+b)X + (a+b+c) = aX^2 + bX + c$$

Comparing coefficients:

- X^2: $a = a$ (Always true)
- X^1: $2a + b = b \implies 2a = 0 \implies a = 0$
- X^0: $a + b + c = c \implies 0 + b + c = c \implies b = 0$

Thus, $a = 0$ and $b = 0$. c is free.

$$\text{Ker}(f) = \{c \cdot 1 \mid c \in \mathbb{R}\} = \text{span}\{1\}$$

The kernel consists of all constant polynomials.

3. **Determine the image of f.**
   We find the image of the basis vectors $\mathcal{B} = (1, X, X^2)$:

- $f(1) = 1 - 1 = 0$
- $f(X) = (X+1) - X = 1$
- $f(X^2) = (X+1)^2 - X^2 = (X^2 + 2X + 1) - X^2 = 2X + 1$

The image is spanned by $\{1, 2X+1\}$. Since these are linearly independent, they form a basis for the image.

$$\mathrm{Im}(f) = \mathrm{span}\{1, X\} = \mathbb{R}_1[X]$$

(The space of polynomials of degree ≤ 1).

4. Compute the matrix of f in the basis $\mathcal{B}$.

We write the coordinates of the images $f(1), f(X), f(X^2)$ as columns relative to the basis $(1, X, X^2)$.

- $f(1) = 0 \implies \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
- $f(X) = 1 = 1 \cdot 1 + 0 \cdot X + 0 \cdot X^2 \implies \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
- $f(X^2) = 1 + 2X = 1 \cdot 1 + 2 \cdot X + 0 \cdot X^2 \implies \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

The matrix is:

$$\mathrm{Mat}_{\mathcal{B}}(f) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$