Chapter 6

Differential Equations

Differential equations are the best way to describe most engineering, mathematical and scientific issues alike, such as describing heat transfer processes or fluid flow, wave motion and electronic circuits and using them in issues of the structural structures of matter or the mathematical description of chemical reactions.

6.1 Basic Concepts

This chapter includes a set of definitions and concepts in differential equations, the most important of which are:

Definition 6.1

A differential equation is every equation that contains differentials or derivatives of one or more functions with respect to variables and is of the form:

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0. \qquad (E)$$

Example 6.1

$$\frac{dx}{dy}z + ydx = u$$

The differential equation is classified into: 1- Ordinary differential equation: It is a differential equation that contains derivatives or ordinary differentials of one or more variables.

Example 6.2

$$ydx + xdy = e^z.$$

2- Partial differential equation: It is a differential equation that contains derivatives or partial differentials of one or more variables.

Example 6.3

$$\frac{\partial x}{\partial y} = zx.$$

3- For the linear ordinary differential equation: it is the equation that is linear with respect to each of the function(s) and their derivatives does not contain their products.

4- Linear partial differential equation: It is the equation that is linear with respect to the partial derivatives of the existing function or functions.

Remark 6.1.1. 1- *The order of an equation is the order of the highest derivative present in it.*

2- *The differential equation can be converted from one form to another to facilitate its solution.*

6.1.1 Order and Degree

Order

Definition 6.2

The order of a differential equation: is the order of the highest derivative (also known as differential coefficient) present in the equation.

Example 6.4

$$\frac{dy}{dx} + y^3 = \cos(x)$$

Contains only the first derivative $\frac{dy}{dx}$, which is a first order differential equation.

Example 6.5

$$\frac{d^3x}{dx^3} + 3x\frac{dy}{dx} = e^y$$

In this equation, the order of the highest derivative is 3 hence, this is a third order differential equation.

Degree

Definition 6.3

The degree of the differential equation is represented by the power of the highest order derivative in the given differential equation.

The differential equation must be a polynomial equation in derivatives for the degree to be defined.

Example 6.6

$$\left(\frac{dy}{dx}\right)^4 + \left(\frac{d^2y}{d^2x}\right)^3 + y = \cos(x)$$

Here, the up exponent is of the derivative of the highest order derivative is 2 and the given differential equation is a polynomial equation in derivatives. So it is a Second Order Three Degree ordinary differential equation

6.1.2 Initial and Final Conditions

In the problems you are required to check the solution of the ordinary differential equation, you can also find the optional constants that appear in the general solution to the equation, and this is done through the initial conditions that are given at the beginning.

In the event that there is a general solution to a differential equation of the second order, for example, that contains two optional constants, two additional conditions for the equation are required to determine the constants. If the two conditions are given at two different points $y(x_1) = y_1$ and $y(x_2) = y_2$, then the conditions are boundary conditions, and the differential equation is called in addition to the boundary conditions: the issue of value limitation.

6.2 Solving Differential Equations

A Differential Equation can be a very natural way of describing something. But it is not very useful as it is. We need to solve it! We solve it when we discover the function y (or set of functions y) that satisfies the equation, and then it can be used successfully.

Definition 6.4

We call the function $y = y(x)$ a solution to the differential equation:

$$F(x, y, y', y'', ..., y^n)$$

if 1- is n times differentiable. 2- checks the differential equation i.e.:

$$F(x, y(x), y'(x), ..., y^{(n)}(x)) = 0$$

So let's look at some different types of differential equations and how to solve them:

6.2.1 Separation of Variables

Separation of variables is a special method to solve some differential equations,

When can I use it?

Separation of Variables can be used when: all the y terms (including dy) can be moved to one side of the equation, and all the x terms (including dx) to the other side.

Example 6.7

In this example, we will explain the stages of solving a differential equation by separating the variables.

$$\frac{dy}{dx} = ky$$

Step 1: Separate the variables by moving all the y terms to one side of the equation and all the x terms to the other side:

$$\frac{dy}{y} = kdx$$

Step 2: Integrate both sides of the equation separately.

$$\int \frac{dy}{y} = \int kdx \Longrightarrow \ln(y) + C = kx + D$$

C is the constant of integration. And we use D for the other, as it is a different constant. Step 3 Simplify: We can roll the two constants into one ($a = D - C$)

$$\ln(y) = kx + a \Rightarrow y = ce^{kx}, \quad (c = e^a)$$

This type of differential equations are of the first order, appearing in many real-world examples.

Example 6.8

Solve the following differential equation:

$$xy^2dx + (1 - x^2)dy = 0.$$

Solution: We divide both sides of the equation by $y^2(1 - x^2)$, so we get:

$$\frac{xdx}{1 - x^2} + \frac{dy}{y^2} = 0$$

Which is a differential equation that can separate the variables and the way to solve it is as follows: By integrating the two sides

$$\begin{aligned} \int \frac{xdx}{1 - x^2} + \int \frac{dy}{y^2} &= 0 \Rightarrow -\frac{1}{2}\ln\left(x^2 - 1\right) - \frac{1}{y} = c \\ &\Rightarrow \ln\left(x^2 - 1\right)^{-\frac{1}{2}} - \frac{1}{y} = c \\ &\Rightarrow \frac{1}{y} = \ln\left(x^2 - 1\right)^{-\frac{1}{2}} - c, \end{aligned}$$

So the solution to the differential equation is

$$y = \left(\ln\left(x^2 - 1\right)^{-\frac{1}{2}} - c\right)^{-1}.$$

6.2.2 Linear Differential Equation

Definition 6.5

The differential equation is linear if the dependent variable and its derivatives in the equation are of first degree. The general form of a linear differential equation of the first order is:

$$\frac{dy}{dx} + yP(x) = Q(x)$$

It is called linear in y. As for the linear equation in x, it takes the form:

$$\frac{dx}{dy} + xa(y) = b(y)$$

The general solution of the differential equation of the first order is of the form:

$$y(x) = e^{-I(x)} \left(\int e^{I(x)} Q(x)\, dx + c \right)$$

where :

$$I(x) = \int P(x)\, dx$$

and c is a constant.

Example 6.9

Find the general solution to the following differential equation:

$$(y + y^2)dx - (y^2 + 2xy + x)dy = 0$$

The solution : The equation is linear in x, so it can be put in the following form:

$$\frac{dx}{dy} + xa(y) = b(y)$$

Dividing both sides of the equation by $dy(y + y^2)$, we get

$$\frac{dx}{dy} - \frac{y^2 + 2xy + x}{y + y^2} = 0$$

so that

$$\frac{dx}{dy} - \frac{y^2}{y + y^2} - \frac{2xy + x}{y + y^2} = 0 \Longrightarrow \frac{dx}{dy} - \frac{2y + 1}{y + y^2}x = \frac{y^2}{y + y^2}$$

By comparing the resulting equation with the first equation, we find

$$b(y) = \frac{y^2}{y + y^2}, \qquad a(y) = -\frac{2y + 1}{y + y^2}$$

Then

$$I\left(y\right) = e^{-\int \frac{2y+1}{y+y^2}dy} = e^{\ln\left(\frac{1}{y+y^2}\right)} = e^{-\ln\left(y+y^2\right)} = \frac{1}{y+y^2}$$

and

$$\int I\left(y\right) b\left(y\right) dy = \int \frac{1}{y+y^2}\frac{y}{y+1}dy = \int \frac{1}{\left(y+1\right)^2}dy = -\frac{1}{y+1}$$

be the solution of the equation

$$I\left(y\right) x = \int I\left(y\right) b\left(y\right) dy + c$$

$$\frac{1}{y+y^2}x = -\frac{1}{y+1} + c$$

so

$$x = -y + c(y^2 + y), c \in \mathbb{R}$$

It is the general solution to the differential equation.

6.2.3 Homogeneous Equations

A first order Differential Equation is Homogeneous when it can be in this form:

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

We can solve it using separation of variables but first we create a new variable $v = \dfrac{y}{x}$. Using $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ we can solve the differential equation. This example shows how this is done.

Example 6.10

Solve

$$\frac{dy}{dx} = \frac{x^2+y^2}{xy}$$

Can we get it in $F(\frac{y}{x})$ style? We have:

$$\begin{aligned} \frac{x^2+y^2}{xy} &= \frac{x}{y} + \frac{y}{x} \\ &= \left(\frac{y}{x}\right)^{-1} + \frac{y}{x} \end{aligned}$$

So

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^{-1} + \frac{y}{x}.$$

Now use separation of variables

$$y = vx \text{ and } \frac{dy}{dx} = v + x\frac{dv}{dx} = v^{-1} + v$$

$$\begin{aligned}
&\Longrightarrow \quad x\frac{dv}{dx} = v^{-1}\\
&\Longrightarrow \quad vdv = \frac{1}{x}dx\\
&\Longrightarrow \quad \frac{v^2}{2} = \ln x + \ln c\\
&\Longrightarrow \quad v^2 = 2\left(\ln cx\right) \Longrightarrow v = \pm\sqrt{2\left(\ln cx\right)}
\end{aligned}$$

Now substitute back $v = \frac{y}{x}$

$$\frac{y}{x} = \pm\sqrt{2\left(\ln cx\right)} \Rightarrow y = x \pm \sqrt{2\left(\ln cx\right)}.$$

6.2.4 Bernoulli Equation

A Bernoulli differential equation is a type of nonlinear first-order differential equation where the dependent variable appears in the equation in both linear and nonlinear forms. The general form of a Bernoulli differential equation is given by:

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

where $n \neq 1$.

Here is a general outline for solving a Bernoulli differential equation:

Transform the equation into a separable differential equation by dividing both sides by y^n and making the substitution $z = y^{1-n}$.

$$\frac{dz}{dx} + (1-n)p(x)z = (1-n)q(x)$$

This is a first-order linear ordinary differential equation, the solution is in the form:

$$z(x) = e^{-I(x)}\left((1-n)\int e^{I(x)}q\left(x\right)dx + c\right)$$

where :

$$I(x) = (1-n)\int p(x)\,dx$$

and c is a constant.

For y by using the original substitution $z = y^{1-n}$.

$$y = \pm\left(e^{-I(x)}\left((1-n)\int e^{I(x)}q(x)\,dx + c\right)\right)^{\frac{1}{1-n}}$$

Example 6.11

Let the differential equation

$$\frac{dy}{dx} + yx^5 = x^5y^7$$

It is a Bernoulli equation with $P(x) = x^5$, $Q(x) = x^5$, and $n = 7$, let's try the substitution:

$$u = y^{1-n} = y^{-6}$$

In terms of y that is:

$$y = u^{-1/6}$$

Differentiate y with respect to x:

$$\frac{dy}{dx} = \frac{-1}{6}u^{-7/6}\frac{du}{dx}.$$

Substitute $\frac{dy}{dx}$ and y into the original equation

$$\frac{-1}{6}u^{-7/6}\frac{du}{dx} + x^5u^{-1/6} = x^5u^{-7/6}$$

Multiply all terms by $-6u^{7/6}$

$$\frac{du}{dx} + x^5u = -6x^5.$$

We now have an equation we can hopefully solve.

6.2.5 Riccati Equation

A Riccati differential equation is a nonlinear first-order differential equation used in control theory, dynamic systems analysis, and system theory.

One way to solve a Riccati differential equation is by transforming it into a linear differential equation using a change of variables. This is often referred to as the "substitution method".

Here is a general outline of the steps involved:

Make a substitution of the form: $y = u'/u$, where u is a new variable.

Substitute y into the Riccati differential equation to obtain a linear differential equation in terms of u.

Solve the linear differential equation for u using standard techniques, such as separation of variables or the method of undetermined coefficients.

Once u has been found, y can be found using the original substitution $y = u'/u$.

Finally, the general solution to the original Riccati differential equation can be found by integrating y to find a function for u, and then finding u' from y.

A Riccati equation has this form:

$$\frac{dy}{dx} + P(x)\,y^2 + q(x)\,y + r(x) = 0 \qquad (1)$$

If $p(x) = 0$; then equation (1) is linear; If $r(x) = 0$; then equation (1) is Bernoulli; If p; q and r are constants, then equation (1) is separable

$$\frac{dy}{py^2 + qy + r} = dx$$

Example 6.12

Solve the differential equation

$$y' = y + y^2 + 1.$$

The given equation is a simple Riccati equation with constant coefficients. Here the variables x and y can be easily separated, so the general solution of the equation is given by

$$\begin{aligned}
\frac{dy}{dx} &= y + y^2 + 1, \;\Rightarrow \frac{dy}{y + y^2 + 1} = dx, \\
&\Rightarrow \int \frac{dy}{y + y^2 + 1} = \int dx, \\
&\Rightarrow \int \frac{dy}{y^2 + y + \frac{1}{4} + \frac{3}{4}} = \int dx, \\
&\Rightarrow \int \frac{dy}{\left(y + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \int dx, \\
&\Rightarrow \frac{1}{\frac{\sqrt{3}}{2}} \arctan \frac{y + \frac{1}{2}}{\frac{\sqrt{3}}{2}} = x + C, \\
&\Rightarrow \frac{2}{\sqrt{3}} \arctan \frac{2y + 1}{\sqrt{3}} = x + C.
\end{aligned}$$

6.2.6 Second Order Equation

We can solve a second order differential equation of the type:

$$\boxed{\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = f(x)}$$

where $P(x)$, $Q(x)$ and $f(x)$ are functions of x, by using:

- Undetermined coefficients which only works when $f(x)$ is a polynomial, exponential, sine, cosine or a linear combination of those.
- Variation of parameters which works on a wide range of functions.

Here we begin by learning the case where $f(x) = 0$ (this makes it "homogeneous"):

$$\boxed{\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0.}$$

and also where the functions $P(x)$ and $Q(x)$ are constants a and b:

$$\boxed{\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0.}$$

We are going to use a special property of the derivative of the exponential function: At any point the slope (derivative) of e^x equals the value of e^x : And when we introduce a value r like this:

$$f(x) = e^{rx}.$$

We find:

$$f'(x) = re^{rx} \quad and \quad f'(x) = r^2e^{rx}$$

In other words, the first and second derivatives of $f(x)$ are both multiples of $f(x)$. This is going to help us a lot!

Theorem 6.1

Let the differential equation

$$\boxed{\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = Q(x)}$$

and let $\Delta = a^2 - 4b$ be the discriminant of the characteristic equation of her

$$\boxed{r^2 + ar + b = 0}$$

−(1 If $\Delta > 0$ and r_1 and r_2 are roots of the characteristic equation, the general solution is:

$$\boxed{y = C_1 e^{r_1 x} + C_2 e^{r_2 x} + y_p(x)}$$

where C_1 and C_2 are constants and $y_p(x)$ a particular solution.

−(2 If $\Delta = 0$ and r is a double root of the characteristic equation, then the general solution is:

$$\boxed{y = e^{rx}\left(C_1 + C_2 x\right) + y_p(x)}$$

where C_1 and C_2 are constants and $y_p(x)$ a particular solution.

−(3 If $\Delta < 0$ and $r = \alpha + i\beta$ is a root of the characteristic equation, then the general solution is:

$$\boxed{y = e^{\alpha x}\left(C_1 \cos\left(\beta x\right) + C_2 \sin\left(\beta x\right)\right) + y_p(x)}$$

where C_1 and C_2 are constants and $y_p(x)$ a particular solution.

Example 6.13

Let the equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

Let $y = e^{rx}$ so we get:

$$\frac{dy}{dx} = re^{rx} \quad \text{and} \quad \frac{d^2y}{dx^2} = r^2 e^{rx}$$

Substitute these into the equation above:

$$r^2 e^{rx} + re^{rx} - 6e^{rx} = 0$$

Simplify:

$$e^{rx}(r^2 + r - 6) = 0 \Longrightarrow r^2 + r - 6 = 0.$$

We have reduced the differential equation to an ordinary quadratic equation! This quadratic equation is given the special name of characteristic equation. We can factor this one to:

$$(r-2)(r+3) = 0 \Longrightarrow r_1 = 2 \quad \text{and} \quad r_2 = -3$$

and so we have two solutions:

$$y_1 = e^{2x} \text{ and } y_2 = e^{-3x}$$

But that's not the final answer because we can combine different multiples of these two answers to get a more general solution:

$$y = Ay_1 + By_2 = Ae^{2x} + Be^{-3x}$$

Example 6.14

$$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0$$

The characteristic equation is:

$$4r^2 + 4r + 1 = 0$$

Then

$$(2r+1)^2 = 0 \Longrightarrow r = -\frac{1}{2}$$

So the solution of the differential equation is:

$$y = Ae^{(\frac{1}{2})x} + Bxe^{(-\frac{1}{2})x} = (A + Bx)\,e^{(-\frac{1}{2})x}$$

Example 6.15

$$(I): \quad \frac{d^2y}{d^2x} + y = 2$$

The characteristic equation is:

$$r^2 + 1 = 0$$

then

$$r_1 = i \quad \text{and} \quad r_2 = -i.$$

So the solution is in the form:

$$\boxed{y = C_1e^{it} + C_2e^{-it} + y_p(t)}$$

Where $y_p(t)$ is the special solution, and here we want to simplify this amount (put it in another form, which is Euler's formula).

$$\boxed{C_1 e^{it} = C_1\cos(t) + iC_1\sin(t) \text{ and } C_2 e^{-it} = C_2\cos(t) - iC_2\sin(t)}$$

Adding the two equations together (taking into account similar terms), we get:

$$\boxed{y = C_1 e^{it} + C_2 e^{-it} = (C_1 + C_2)\cos(t) + i(C_1 - C_2)\sin(t)}$$

And by substituting in the equation (I), that is, the special solution is equal to 2, so that the equation becomes:

$$y = A\cos(t) + B\sin(t) + 2,$$

this is the general solution to the equation.

6.2.7 Particular Solution

Particular solution of the differential equation is a unique solution of the form y = f(x), which satisfies the differential equation. The particular solution of the differential equation is derived by assigning values to the arbitrary constants of the general solution of the differential equation.

$$\boxed{\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = f(x)}$$

$f(x)$ may include both sine and cosine functions. However, even if $f(x)$ included a sine term only or a cosine term only, both terms must be present in the guess. The method of undetermined coefficients also works with products of polynomials, exponentials, sines, and cosines. Some of the key forms of $f(x)$ and the associated guesses for $y_p(x)$ are summarized in this Table.

$f(x)$	Initial guess for $y_p(x)$
k (a constant)	A (a constant)
$ax+b$	$Ax+B$ The guess must include both terms even if $b=0$.
ax^2+bx+c	Ax^2+Bx+C The guess must include all three terms even if b or c are zero.
Higher-order polynomials	Polynomial of the same order as $f(x)$
$ae^{\lambda x}$	$Ae^{\lambda x}$
$ae^{\alpha x}\cos\beta x + be^{\alpha x}\sin\beta x$	$Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x$
$(ax^2+bx+c)e^{\lambda x}$	$(Ax^2+Bx+C)e^{\lambda x}$
$(a_2x^2+a_1x+a_0)\cos\beta x$ $+(b_2x^2+b_1x+b_0)\sin\beta x$	$(A_2x^2+A_1x+A_0)\cos\beta x$ $+(B_2x^2+B_1x+B_0)\sin\beta x$
$(a_2x^2+a_1x+a_0)e^{\alpha x}\cos\beta x$ $+(b_2x^2+b_1x+b_0)e^{\alpha x}\sin\beta x$	$(A_2x^2+A_1x+A_0)e^{\alpha x}\cos\beta x$ $+(B_2x^2+B_1x+B_0)e^{\alpha x}\sin\beta x$

Example 6.16

Let

$$\frac{dy}{dx} = x^2 \Longrightarrow dy = x^2 dx$$

Integrating both sides, we get

$$\int dy = \int x^2 dx$$

If we solve this equation to figure out the value of y we get

$$y = \frac{x^3}{3} + C$$

where C is an arbitrary constant. In the above-obtained solution, we see that y is a function of x. On substituting this value of y in the given differential equation, both the sides of the differential equation becomes equal.

6.3 Exercise series N° 6