

Tutorial Series N°02: Linear Maps

Exercise 1. Determine whether the following mappings are linear:

- a) $T_1 : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $T_1(x, y, z) = (x + y, y - z)$.
- b) $T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $T_2(x, y) = xy$.
- c) $T_3 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T_3(x, y, z) = (x + 1, y, z)$.

Exercise 2. Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear map defined by:

$$f(x, y, z) = (x + y + z, \quad 2x - z, \quad 3x + y)$$

1. Determine $\text{Ker}(f)$. Is f injective?
2. Determine a basis for $\text{Im}(f)$. Is f surjective?
3. Verify the Rank-Nullity Theorem ($\dim(\mathbb{R}^3) = \dim(\text{Ker } f) + \dim(\text{Im } f)$).

Exercise 3. Let $T : \mathbb{R}^5 \rightarrow \mathbb{R}^3$ be a linear map such that $\text{rank}(T) = 2$.

1. Calculate $\dim(\text{Ker } T)$ (the nullity).
2. Can T be injective? Justify your answer.
3. Can T be surjective? Justify your answer.

Exercise 4. Let $f_m : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be an endomorphism depending on a parameter $m \in \mathbb{R}$, defined by:

$$f_m(x, y, z) = (x + y + z, \quad x + my + z, \quad x + y + mz)$$

1. Find the kernel $\text{Ker}(f_m)$ depending on the values of m .
2. For which values of m is f_m an automorphism (bijective)?
3. For $m = 1$, find the image $\text{Im}(f_1)$ and its dimension.

Exercise 5. Let

$$f : \mathbb{R}_2[X] \longrightarrow \mathbb{R}_2[X], \quad P(X) \longmapsto P(X + 1) - P(X),$$

and let $\mathcal{B} = (1, X, X^2)$ be the canonical basis of $\mathbb{R}_2[X]$.

1. Verify that f is a linear map.
2. Determine the kernel of f .
3. Determine the image of f .
4. Compute the matrix of f in the basis \mathcal{B} .
5. Let $\mathcal{B}' = (1, X - 1, (X - 1)(X - 2))$. Show that \mathcal{B}' is a basis of $\mathbb{R}_2[X]$ and compute $\text{Mat}_{\mathcal{B}'}(f)$.

Good Luck!