

Solutions: Tutorial Series N°01

Solution to Exercise 01

a) F_1 : **YES**. It is a plane passing through the origin.

- $0 + 0 - 0 = 0 \implies 0_E \in F_1$.
- If $u, v \in F_1$, then $(u + v)$ satisfies the linear homogeneous equation.

b) F_2 : **NO**.

- The zero vector $(0, 0, 0)$ is not in F_2 because $0 + 0 = 0 \neq 1$.

c) F_3 : **NO**.

- Let $u = (1, 0, 0)$ and $v = (0, 1, 0)$. Both are in F_3 since $xy = 0$.
- However, $u + v = (1, 1, 0)$. Here $x \cdot y = 1 \cdot 1 = 1 \neq 0$.
- Not closed under addition.

d) F_4 : **YES**. It is the intersection of two planes (a line) passing through the origin.

e) F_5 : **NO**.

- Let $u = (1, 0, 0) \in F_5$.
- Let $\lambda = -1$. Then $\lambda u = (-1, 0, 0)$. Since $x = -1 < 0$, $\lambda u \notin F_5$.
- Not closed under scalar multiplication.

f) F_6 : **YES**.

- In \mathbb{R} , $x^2 + y^2 = 0 \implies x = 0$ and $y = 0$.
- So $F_6 = \{(0, 0, z) \mid z \in \mathbb{R}\}$. This is the z -axis, which is a subspace.

Solution to Exercise 02

1. **Proving it is a Basis:** We calculate the determinant of the matrix formed by the vectors:

$$\det(v_1, v_2, v_3) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix}$$

Expanding along the last row:

$$\begin{aligned} &= 1 \begin{vmatrix} 2 & 3 \\ 9 & 3 \end{vmatrix} - 0 + 4 \begin{vmatrix} 1 & 2 \\ 2 & 9 \end{vmatrix} \\ &= 1(6 - 27) + 4(9 - 4) = -21 + 20 = -1 \end{aligned}$$

Since $\det \neq 0$, the vectors are linearly independent. As $\dim(\mathbb{R}^3) = 3$, they form a basis.

2. **Finding Coordinates:** We solve $\alpha v_1 + \beta v_2 + \gamma v_3 = u$.

$$\begin{cases} \alpha + 2\beta + 3\gamma = 5 \\ 2\alpha + 9\beta + 3\gamma = 14 \\ \alpha + 4\gamma = 5 \end{cases}$$

From (3), $\alpha = 5 - 4\gamma$. Substitute into (1): $(5 - 4\gamma) + 2\beta + 3\gamma = 5 \implies 2\beta - \gamma = 0 \implies \gamma = 2\beta$. Substitute into (2): $2(5 - 4(2\beta)) + 9\beta + 3(2\beta) = 14 \implies 10 - 16\beta + 9\beta + 6\beta = 14 \implies 10 - \beta = 14 \implies \beta = -4$. Then $\gamma = -8$ and $\alpha = 5 - 4(-8) = 37$.

$$u = 37v_1 - 4v_2 - 8v_3 \implies [u]_{\mathcal{B}} = (37, -4, -8)$$

For $w = (0, -4, 5)$, solving the system yields:

$$[w]_{\mathcal{B}} = (1, -2, 1)$$

Solution to Exercise 03

1. Linear Dependence: The vectors are dependent if their determinant is zero.

$$\begin{vmatrix} 1 & 2 & 4 \\ -1 & m & 1 \\ 2 & 1 & 5 \end{vmatrix} = 0$$

$$1(5m - 1) - 2(-5 - 2) + 4(-1 - 2m) = 0$$

$$5m - 1 + 14 - 4 - 8m = 0$$

$$-3m + 9 = 0 \implies m = 3$$

The family is dependent if $\mathbf{m} = \mathbf{3}$.

2. Linear Combination (for $\mathbf{m}=1$): We want $v_3 = av_1 + bv_2$.

$$(4, 1, 5) = a(1, -1, 2) + b(2, 1, 1)$$

$$\begin{cases} a + 2b = 4 \\ -a + b = 1 \\ 2a + b = 5 \end{cases}$$

Summing (1) and (2): $3b = 5 \implies b = 5/3$. From (2): $a = b - 1 = 5/3 - 3/3 = 2/3$. Check (3): $2(2/3) + 5/3 = 9/3 = 3 \neq 5$. *Correction:* Wait, for $m = 1$, the determinant is $-3(1) + 9 = 6 \neq 0$. They are independent! Therefore, v_3 **cannot** be written as a combination of v_1, v_2 . (If the question implied $m = 3$, then $v_3 = 2v_1 + 1v_2$).

Solution to Exercise 04

1. Dimensions: F : Vectors $(1, 1, 0)$ and $(0, 1, 1)$ are not proportional. $\dim(F) = 2$. G : Generated by one non-zero vector. $\dim(G) = 1$.

2. Sum $F + G$: We form a matrix with the 3 vectors and check rank.

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \det(M) = 1(1) - 0 + 1(1) = 2 \neq 0$$

Since the 3 vectors are independent, they span \mathbb{R}^3 . Thus $F + G = \mathbb{R}^3$.

3. Direct Sum: Using the dimension theorem: $\dim(F + G) = \dim(F) + \dim(G) - \dim(F \cap G)$. $3 = 2 + 1 - \dim(F \cap G) \implies \dim(F \cap G) = 0$. Since the intersection is $\{0\}$, the sum is **Direct**.

Solution to Exercise 05

1. **Basis Check:** Coordinate matrix in standard basis $\{1, X, X^2\}$:

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$\det(M) = 1(-1) - 1(1) + 0 = -2 \neq 0$. The polynomials are independent and form a basis.

2. **Coordinates:** For $Q(X) = 1 + X + X^2$: Looking at the sum: $P_1 + P_2 + P_3 = (1 + X) + (1 + X^2) + (X + X^2) = 2 + 2X + 2X^2 = 2Q(X)$. So $Q(X) = \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3$.

For $R(X) = 2 - X + 3X^2$: Solve $a(1 + X) + b(1 + X^2) + c(X + X^2) = 2 - X + 3X^2$.

$$\begin{cases} a + b = 2 \\ a + c = -1 \\ b + c = 3 \end{cases}$$

(3)-(2): $b - a = 4$. Add to (1): $2b = 6 \implies b = 3$. Then $a = -1$, $c = 0$. $R(X) = -1P_1 + 3P_2$.