

# Chapter 1

## Vector Spaces (Espaces Vectoriels)

### 1.1 Algebraic Structures: Background

To rigorously define a vector space, we first recall the definition of a **Field**.

#### Definition 1.1: Field (Corps)

A set  $\mathbb{K}$  equipped with two operations  $(+, \times)$  is a field if:

- $(\mathbb{K}, +)$  is an abelian group (neutral element 0).
- $(\mathbb{K} \setminus \{0\}, \times)$  is an abelian group (neutral element 1).
- Multiplication is distributive over addition.

Common examples:  $\mathbb{R}$  (Real numbers),  $\mathbb{C}$  (Complex numbers),  $\mathbb{Q}$  (Rationals).

### 1.2 Definition of a Vector Space

#### Definition 1.2: Vector Space

Let  $\mathbb{K}$  be a field. A non-empty set  $E$  is a **Vector Space over**  $\mathbb{K}$  if it is equipped with:

1. An **internal composition law** (Addition):

$$+ : E \times E \rightarrow E, \quad (u, v) \mapsto u + v$$

2. An **external composition law** (Scalar Multiplication):

$$\cdot : \mathbb{K} \times E \rightarrow E, \quad (\lambda, u) \mapsto \lambda \cdot u$$

These operations must satisfy the following **8 Axioms**:

- **(V1) Associativity:**  $\forall u, v, w \in E, (u + v) + w = u + (v + w)$ .
- **(V2) Commutativity:**  $\forall u, v \in E, u + v = v + u$ .
- **(V3) Neutral Element:**  $\exists 0_E \in E$  such that  $\forall u \in E, u + 0_E = u$ .
- **(V4) Symmetric Element:**  $\forall u \in E, \exists (-u) \in E$  such that  $u + (-u) = 0_E$ .
- **(V5) Distributivity (Scalars):**  $\forall \lambda, \mu \in \mathbb{K}, \forall u \in E, (\lambda + \mu)u = \lambda u + \mu u$ .
- **(V6) Distributivity (Vectors):**  $\forall \lambda \in \mathbb{K}, \forall u, v \in E, \lambda(u + v) = \lambda u + \lambda v$ .
- **(V7) Mixed Associativity:**  $\forall \lambda, \mu \in \mathbb{K}, \forall u \in E, \lambda(\mu u) = (\lambda\mu)u$ .
- **(V8) Unitary Element:**  $1_{\mathbb{K}} \cdot u = u$ .

### 1.2.1 Fundamental Properties (With Proofs)

From the axioms, we can derive essential calculation rules.

#### Proposition 1.1: Calculation Rules

For any  $u \in E$  and  $\lambda \in \mathbb{K}$ :

1.  $0_{\mathbb{K}} \cdot u = 0_E$ .
2.  $\lambda \cdot 0_E = 0_E$ .
3.  $(-1) \cdot u = -u$ .
4.  $\lambda \cdot u = 0_E \implies \lambda = 0_{\mathbb{K}} \text{ or } u = 0_E$  (No zero divisors).

**Proof of 1.** We write  $0_{\mathbb{K}} \cdot u = (0 + 0) \cdot u = 0 \cdot u + 0 \cdot u$ . By adding the inverse  $-(0 \cdot u)$  to both sides, we get  $0_E = 0 \cdot u$ . ■

## 1.3 Vector Subspaces

#### Definition 1.3: Subspace

A subset  $F \subseteq E$  is a subspace if it inherits the vector space structure. Criteria: 1.  $F \neq \emptyset$  (Usually  $0_E \in F$ ). 2.  $\forall u, v \in F, u + v \in F$ . 3.  $\forall \lambda \in \mathbb{K}, u \in F, \lambda u \in F$ .

#### Illustrative Example

**Matrix Subspace:** Let  $E = M_n(\mathbb{R})$ . Let  $F = \{A \in E \mid A^T = A\}$  (Symmetric matrices).

- Zero matrix  $0^T = 0$ , so  $0 \in F$ .
- Let  $A, B \in F$  (so  $A^T = A, B^T = B$ ).  $(A + B)^T = A^T + B^T = A + B$ . So  $A + B \in F$ .
- $(\lambda A)^T = \lambda A^T = \lambda A$ . So  $\lambda A \in F$ .

Conclusion: The set of symmetric matrices is a subspace of  $M_n(\mathbb{R})$ .

## 1.4 Operations on Subspaces

### 1.4.1 Intersection

#### Proposition 1.2: Intersection

If  $\{F_i\}_{i \in I}$  is a family of subspaces, then  $\bigcap_{i \in I} F_i$  is a subspace.

*Remark:* The **Union**  $F \cup G$  is **not** a subspace generally, unless  $F \subset G$  or  $G \subset F$ .

### 1.4.2 Sum of Subspaces

#### Definition 1.4: Sum

The sum of two subspaces  $F$  and  $G$  is the set generated by their union:

$$F + G = \{u + v \mid u \in F, v \in G\}$$

### 1.4.3 Direct Sums and Supplementary Subspaces

This is a critical concept for decomposing spaces.

#### Theorem 1.1: Direct Sum Characterization

The sum  $F + G$  is direct (denoted  $F \oplus G$ ) if and only if any vector  $w \in F + G$  has a **unique** decomposition  $w = u + v$ . Equivalent condition:

$$F \cap G = \{0_E\}$$

#### Definition 1.5: Supplementary

$F$  and  $G$  are supplementary in  $E$  if  $E = F \oplus G$ . This means:

1. Existence:  $\forall x \in E, \exists u \in F, v \in G$  such that  $x = u + v$ .

2. Uniqueness: This decomposition is unique.

### Theorem 1.2: Grassmann's Formula

Let  $F, G$  be finite-dimensional subspaces.

$$\dim(F + G) = \dim(F) + \dim(G) - \dim(F \cap G)$$

If the sum is direct,  $\dim(F \oplus G) = \dim(F) + \dim(G)$ .

## 1.5 Projections and Symmetry

Direct sums define special linear maps.

### Definition 1.6: Projection on $F$ along $G$

If  $E = F \oplus G$ , any  $x$  writes uniquely as  $x = x_F + x_G$ . The map  $p : E \rightarrow E$  defined by  $p(x) = x_F$  is called the **Projection onto  $F$  parallel to  $G$** . Properties:

- $p$  is linear.
- $p \circ p = p$  ( $p$  is idempotent).
- $\text{Im}(p) = F$  and  $\text{ker}(p) = G$ .

## 1.6 Linear Independence and Bases

### 1.6.1 Span (Linear Hull)

The set of all linear combinations of a family  $S$  is denoted  $\text{Span}(S)$ . It is the smallest subspace containing  $S$ .

### 1.6.2 Linear Independence

#### Definition 1.7: Free Family

A family  $\{v_1, \dots, v_n\}$  is free if the only solution to  $\sum \lambda_i v_i = 0$  is  $\lambda_i = 0 \forall i$ .

**Geometric Interpretation:**

- 2 vectors are dependent  $\iff$  they are collinear (on the same line).
- 3 vectors are dependent  $\iff$  they are coplanar (in the same plane).

### 1.6.3 Basis and Dimension

#### Theorem 1.3: Basis Theorem

If  $E$  has a finite generating set, it admits a basis. All bases have the same cardinality  $n = \dim(E)$ .

#### Illustrative Example

**Basis of Polynomials:** Show that  $\mathcal{B} = \{1, X, X^2, \dots, X^n\}$  is a basis of  $\mathbb{R}_n[X]$ .

1. **Generating:** Any polynomial is by definition a sum  $\sum a_i X^i$ .
2. **Independence:** If  $\sum a_i X^i = 0$  (the zero polynomial), then all coefficients  $a_i$  must be 0.

Thus,  $\dim(\mathbb{R}_n[X]) = n + 1$ .