Chapter 5

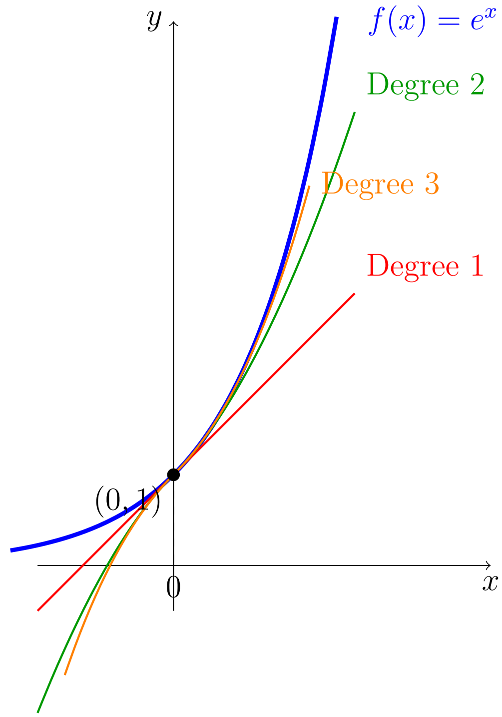
Limited Expansions

5.1 Limited Expansions and Taylor Approximations

Limited expansions provide powerful tools for approximating functions near specific points using polynomials. These approximations become increasingly accurate as we include more terms in the expansion.

5.1.1 Introduction to Limited Expansions

The basic idea is to approximate a function $f(x)$ near a point x_0 using a polynomial that matches the function's value and derivatives at x_0.



5.1.2 Taylor's Theorem

Theorem 5.1 (Taylor's Theorem with Peano Remainder)

Let $f : I \to \mathbb{R}$ be n-times differentiable at $x_0 \in I$. Then there exists a function ε such that:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + (x - x_0)^n \varepsilon(x - x_0)$$

where $\lim_{x \to x_0} \varepsilon(x - x_0) = 0$.

Definition 5.1 (Taylor Polynomial)

The **Taylor polynomial** of degree n for f at x_0 is:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

The remainder term is $R_n(x) = f(x) - P_n(x)$.

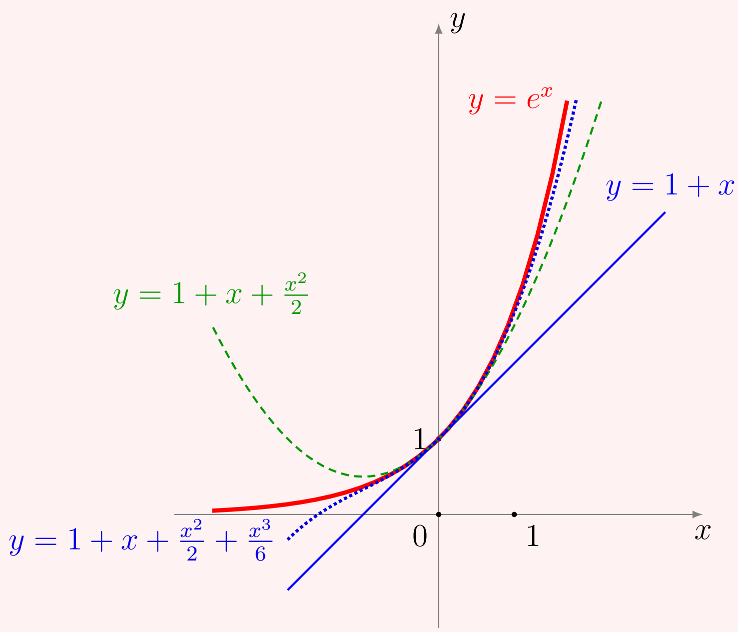
Example 5.1 (Exponential Function)

For $f(x) = e^x$ at $x_0 = 0$:

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= 1 \\ f''(0) &= 1 \\ &\vdots \\ f^{(n)}(0) &= 1 \end{aligned}$$

Thus the Taylor polynomial is:

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$



5.1.3 Maclaurin Series (Special Case at $x_0 = 0$)

When the expansion point is $x_0 = 0$, the Taylor series is called a Maclaurin series.

Theorem 5.2 (Maclaurin Series)

For f infinitely differentiable near 0:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

when the series converges.

Common Maclaurin Expansions

Function	**Maclaurin Expansion**
e^x	$\sum_{k=0}^{\infty} \frac{x^k}{k!}$
$\sin x$	$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$
$\cos x$	$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$
$\ln(1+x)$	$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$
$(1+x)^\alpha$	$\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$
$\frac{1}{1-x}$	$\sum_{k=0}^{\infty} x^k$

Example 5.2 (Sine and Cosine Functions)

For $f(x) = \sin x$:

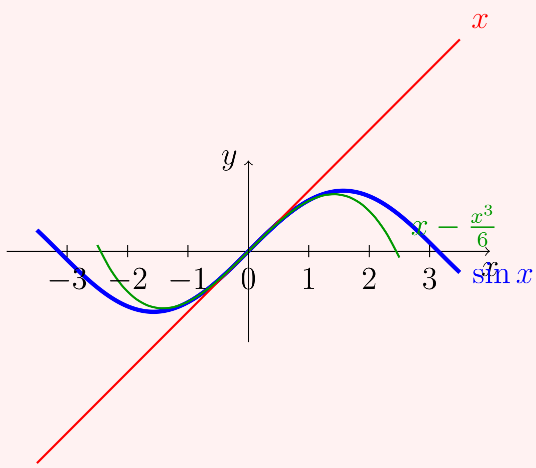
$$\begin{aligned} f(0) &= 0 \\ f'(0) &= 1 \\ f''(0) &= 0 \\ f'''(0) &= -1 \\ f^{(4)}(0) &= 0 \\ &\vdots \end{aligned}$$

Thus:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Similarly for $\cos x$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$



5.1.4 Operations on Limited Expansions

When working with limited expansions, we can perform various operations while maintaining the approximation order.

Proposition 5.1.1 (Algebraic Operations)**.** *Let* $f(x) = P_n(x) + o(x^n)$ *and* $g(x) = Q_n(x) + o(x^n)$ *be limited expansions at* 0*.*

- ***Sum***: $(f + g)(x) = P_n(x) + Q_n(x) + o(x^n)$
- ***Product***: $(f \cdot g)(x) = P_n(x)Q_n(x) + o(x^n)$ *(truncate to degree n)*
- ***Composition***: *If* $g(0) = 0$*, then* $(f \circ g)(x) = P_n(Q_n(x)) + o(x^n)$
- ***Division***: *For* $g(0) \neq 0$, $\frac{1}{g(x)} = \frac{1}{Q_n(x)} + o(x^n)$

Example 5.3 (Product of Expansions)

Compute the limited expansion of $e^x \cos x$ at order 3:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3) \\ \cos x &= 1 - \frac{x^2}{2} + o(x^3) \\ e^x \cos x &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\left(1 - \frac{x^2}{2}\right) + o(x^3) \\ &= 1 + x + \left(\frac{1}{2} - \frac{1}{2}\right)x^2 + \left(\frac{1}{6} - \frac{1}{2}\right)x^3 + o(x^3) \\ &= 1 + x - \frac{x^3}{3} + o(x^3) \end{aligned}$$

5.1.5 Applications and Examples

Example 5.4 (Arctangent Function)

Compute the limited expansion of $\arctan x$ at order 5:

$$\begin{aligned} \frac{d}{dx}\arctan x &= \frac{1}{1+x^2} = 1 - x^2 + x^4 + o(x^5) \\ \arctan x &= \int (1 - x^2 + x^4)dx + o(x^6) \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^6) \end{aligned}$$

Example 5.5 (Tangent Function)

Compute $\tan x$ at order 5 using $\tan x = \frac{\sin x}{\cos x}$:

$$\begin{aligned} \sin x &= x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^6) \\ \cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5) \\ \frac{1}{\cos x} &= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + o(x^5) \\ \tan x &= \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\left(1 + \frac{x^2}{2} + \frac{5x^4}{24}\right) + o(x^5) \\ &= x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^5) \end{aligned}$$

Example 5.6 (Composite Function)

Compute $\sin(\ln(1+x))$ at order 3:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$$
$$\sin u = u - \frac{u^3}{6} + o(u^3)$$
$$\sin(\ln(1+x)) = \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) - \frac{x^3}{6} + o(x^3)$$
$$= x - \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$$

5.1.6 Error Estimation

Theorem 5.3 (Taylor's Remainder Theorem)

If f is $(n+1)$-times differentiable on an interval containing x_0, then for each x in the interval, there exists c between x_0 and x such that:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

Example 5.7 (Error Bound for e^x)

Approximate $e^{0.1}$ using a 3rd degree Taylor polynomial:

$$e^{0.1} \approx 1 + 0.1 + \frac{0.1^2}{2} + \frac{0.1^3}{6} \approx 1.1051667$$

The error satisfies:

$$|R_3(0.1)| \leq \frac{e^{0.1}}{24}(0.1)^4 \leq \frac{1.1052}{24} \times 10^{-4} \approx 4.6 \times 10^{-6}$$

The actual value is $e^{0.1} \approx 1.1051709$, with error $\approx 4.2 \times 10^{-6}$.

5.2 Exercise series N° 3: Real Functions and Limited Expansions