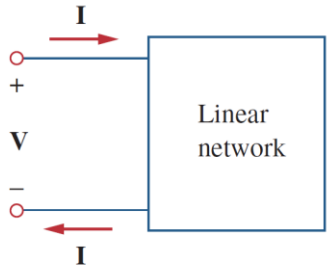
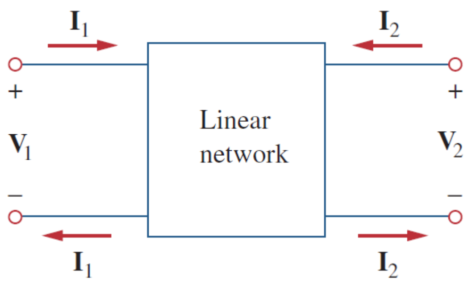
Chapter 2: Passive Two-port Networks 3 weeks

2-1- Definition

A pair of terminals through which a current may enter or leave a network is known as a *port*. Two-terminal devices or elements (such as resistors, capacitors, and inductors) result in one-port networks.



A two-port network is an electrical network with two separate ports for input and output.



The study of two-port networks is for at least two reasons. First, such networks are useful in communications, control systems, power systems, and electronics. For example, they are used in electronics to model transistors and to facilitate cascaded design. Second, knowing the parameters of a two-port network enables us to treat it as a “black box” when embedded within a larger network.

2-2- Two-port network representations

Two-port network representations are mathematical models for analyzing electrical circuits with two pairs of terminals (an input port and an output port). They simplify complex circuits by representing them as a "black box" and relating the port voltages and currents using different parameters, such as Z, Y and h param , or Z-parameter matrix, of a two-port network relates the port voltages (V_1, V_2) to the port currents (I_1, I_2).

2-2-1- Impedance parameters

Impedance and admittance parameters are commonly used in the synthesis of filters. They are also useful in the design and analysis of impedance-matching networks and power distribution networks.

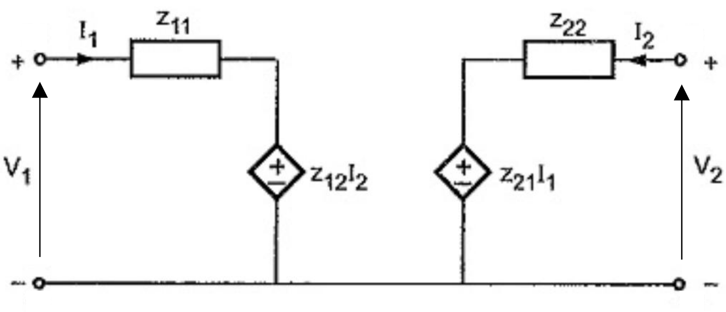
In the impedance model, terminal voltages can be related to the terminal currents through the equation V=Z×I. The matrix is defined as

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$
$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

or in matrix form as

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{z}_{11} & \mathbf{z}_{12} \\ \mathbf{z}_{21} & \mathbf{z}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = [\mathbf{z}] \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

The equivalent network for a two-port network using impedance parameters (z-parameters) can be modeled as a series connection of Z_{11} and Z_{22} with a series current-controlled voltage sources $Z_{12}I_2$ and $Z_{21}I_1$.



Les équations caractéristiques de ce quadripôle peuvent se mettent sous la forme:

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$

$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

ou encore sous forme matricielle:

$$[V] = [Z][I]$$

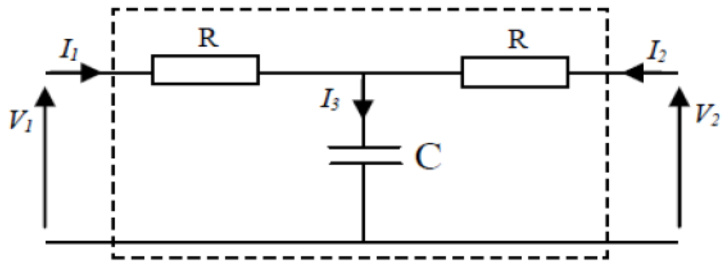
$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = [Z] \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

The values of the parameters can be evaluated by setting $I_1=0$ (input port open-circuited) or $I_2=0$ (output port open-circuited). Thus,

- Open-circuit input impedance: $Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0}$
- Reverse transfer impedance with the input port open: Z_{21}: $Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0}$
- Forward transfer impedance with the output port open: $Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$
- Open-circuit output impedance: $Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$

Example 2.1:

Determine the *z* parameters for the circuit below.



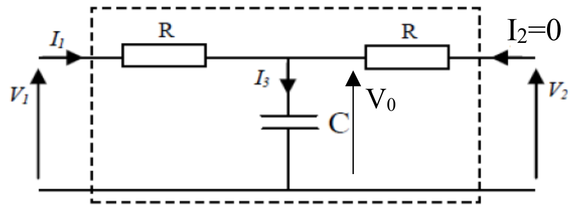
$I_2 = 0 \;\; => \;\; I_3 = I_1$

$=> \; V_1 = (R + Z_C)I_1$

$=> Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = R + Z_C$

$V_2 = V_0 - RI_2 = V_0$

$V_2 = V_0 = Z_C I_1 \;\; => Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = Z_C$



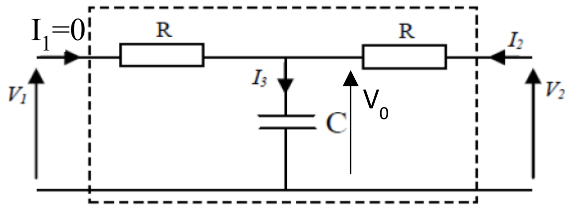
$I_1 = 0 \;\; => \;\; I_3 = I_2$

$=> \; V_2 = (R + Z_C)I_2$

$=> Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = R + Z_C$

$V_1 = V_0 - RI_1 = V_0$

$V_1 = V_0 = Z_C I_2 \;\; => Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = Z_C$



2-2-2- Admittance Parameters:

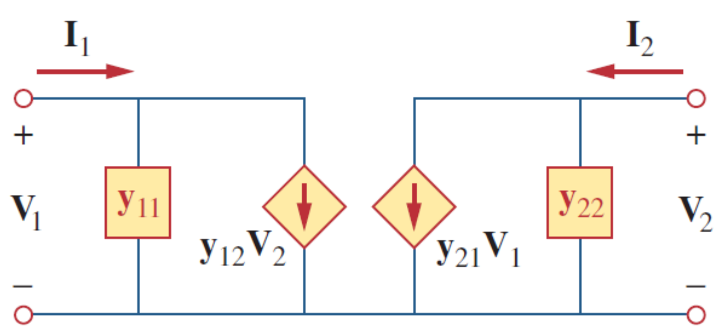
In this case, the terminal currents can be expressed in terms of the terminal voltages as

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$

$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

or in matrix form as

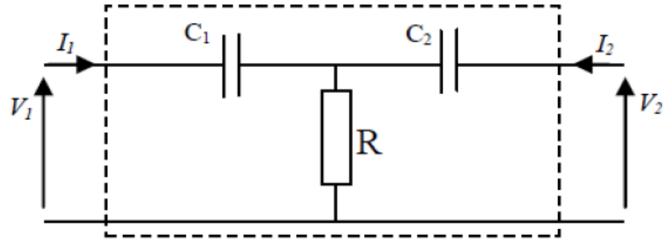
$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = [Y] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$



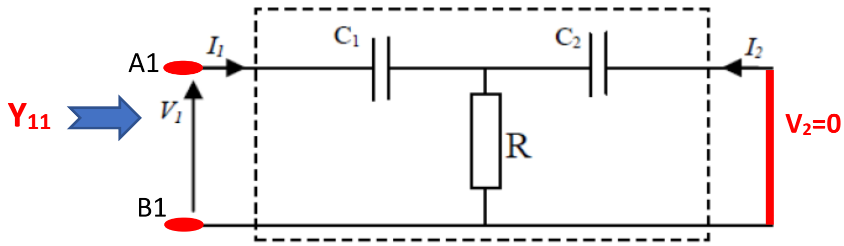
- Short-circuit input admittance: $Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0}$
- Short-circuit transfer admittance from port 2 to port 1: $Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$
- Short-circuit transfer admittance from port 1 to port 2: $Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}$
- Short-circuit output admittance: $Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$

Example 2.2 :

Find the Y parameters of the following high-pass filter.



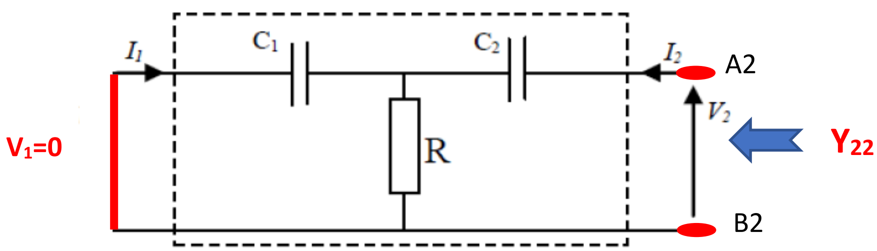
- Input admittance : $Y_{11} = \frac{I_1}{V_1}\Big|_{V_2=0}$



The input admittance is the equivalent admittance seen from the input terminals A1 and B1, hence:

$$Y_{11} = \frac{1}{Z_{A1-B1}} = \frac{1}{Z_{C1} + Z_{C2}//R} = \frac{1}{\frac{1}{j\omega C_1} + \frac{\frac{1}{j\omega C_2}R}{\frac{1}{j\omega C_2} + R}}$$

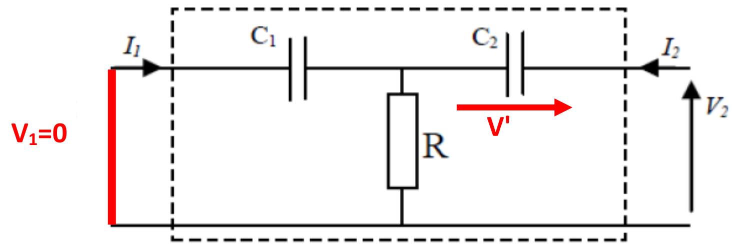
- Output admittance : $Y_{22} = \frac{I_2}{V_2}\Big|_{V_1=0}$



The output admittance is the equivalent admittance seen from the output terminals A2 and B2, thus:

$$Y_{22} = \frac{1}{Z_{A2-B2}} = \frac{1}{Z_{C2} + Z_{C1}//R} = \frac{1}{\frac{1}{j\omega C_2} + \frac{\frac{1}{j\omega C_1}R}{\frac{1}{j\omega C_1} + R}}$$

- Short-circuit reverse transfer admittance: $Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$



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Voltage divider: $V' = \frac{Z_{C2}}{Z_{C2}+R//Z_{C1}} V_2$ (1)

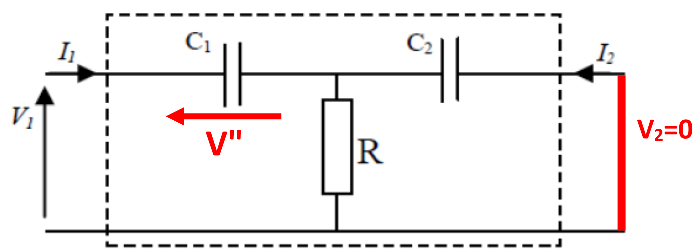
Ohm's law: $V' = Z_{C2} I_2$ (2)

Current divider: $I_1 = \frac{\frac{1}{Z_{C1}}}{\frac{1}{Z_{C1}}+\frac{1}{R}} I_2$ (3)

By substituting (1) and (2) into (3), we obtain:

$$I_1 = \frac{\frac{1}{Z_{C1}}}{\frac{1}{Z_{C1}}+\frac{1}{R}} \frac{1}{Z_{C2}} \frac{Z_{C2}}{Z_{C2}+R//Z_{C1}} V_2 \Rightarrow Y_{12} = \frac{I_1}{V_2}\Big|_{V_1=0} = \frac{\frac{1}{Z_{C1}}}{\frac{1}{Z_{C1}}+\frac{1}{R}} \frac{1}{Z_{C2}} \frac{Z_{C2}}{Z_{C2}+R//Z_{C1}}$$

- Short-circuit forward transfer admittance: $Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}$



Voltage divider: $V'' = \frac{Z_{C1}}{Z_{C1}+R//Z_{C2}} V_1$ (4)

Ohm's law: $V'' = Z_{C1} I_1$ (5)

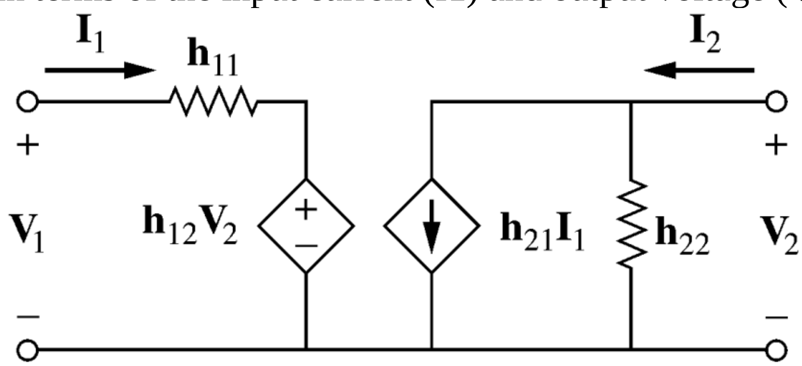
Current divider: $I_2 = \frac{\frac{1}{Z_{C2}}}{\frac{1}{Z_{C2}}+\frac{1}{R}} I_1$ (6)

By substituting (4) and (5) into (6), we obtain:

$$I_2 = \frac{\frac{1}{Z_{C2}}}{\frac{1}{Z_{C2}}+\frac{1}{R}} \frac{1}{Z_{C1}} \frac{Z_{C1}}{Z_{C1}+R//Z_{C2}} V_1 \Rightarrow Y_{21} = \frac{I_2}{V_1}\Big|_{V_2=0} = \frac{\frac{1}{Z_{C2}}}{\frac{1}{Z_{C2}}+\frac{1}{R}} \frac{1}{Z_{C1}} \frac{Z_{C1}}{Z_{C1}+R//Z_{C2}}$$

2-2-3- Hybrid parameters:

The *z* and *y* parameters of a two-port network do not always exist. So, there is a need for developing another set of parameters. Hybrid parameters, or h-parameters, for a two-port network describe the relationship between the input voltage (V1) and output current (I2) in terms of the input current (I1) and output voltage (V2**).**



$$\mathbf{V}_1 = \mathbf{h}_{11}\mathbf{I}_1 + \mathbf{h}_{12}\mathbf{V}_2$$
$$\mathbf{I}_2 = \mathbf{h}_{21}\mathbf{I}_1 + \mathbf{h}_{22}\mathbf{V}_2$$

The values of the parameters are determined as

$$\mathbf{h}_{11} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_1}\right|_{\mathbf{V}_2=0}, \qquad \mathbf{h}_{12} = \left.\frac{\mathbf{V}_1}{\mathbf{V}_2}\right|_{\mathbf{I}_1=0}$$

$$\mathbf{h}_{21} = \left.\frac{\mathbf{I}_2}{\mathbf{I}_1}\right|_{\mathbf{V}_2=0}, \qquad \mathbf{h}_{22} = \left.\frac{\mathbf{I}_2}{\mathbf{V}_2}\right|_{\mathbf{I}_1=0}$$

It is evident that the parameters h_{ij} represent an impedance, a voltage gain, a current gain, and an admittance, respectively. This is why they are called the hybrid parameters.

$\mathbf{h}_{11}$ = Short-circuit input impedance

$\mathbf{h}_{12}$ = Open-circuit reverse voltage gain

$\mathbf{h}_{21}$ = Short-circuit forward current gain

$\mathbf{h}_{22}$ = Open-circuit output admittance

The procedure for calculating the *h* parameters is similar to that used for the *z* or *y* parameters.

2-3- Passive filters

Frequency-selective or filter circuits pass to the output only those input signals that are in a desired range of frequencies (called pass band). The amplitude of signals outside this range of frequencies (called stop band) is reduced (ideally reduced to zero).

The main parameter is the voltage transfer function in the frequency domain,

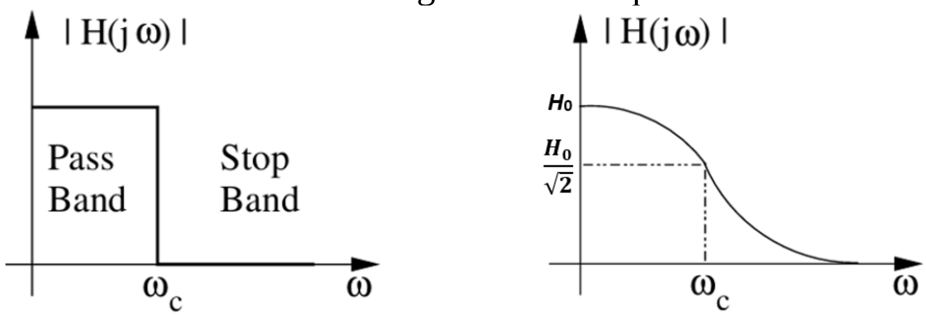
$$H_v(j\omega) = V_o/V_i$$

Where Vi and Vo are the input and output voltage of the two-port network respectively.

2-3-1- Low-pass filter

A passive low-pass filter (LPF) is a two-port network composed of passive components, such as resistors (R), capacitors (C), and/or inductors (L), that works without an external power source.

- It allows signals with frequencies below a specified "cutoff frequency" ω_c to pass through with minimal attenuation.
- It blocks or attenuates signals with frequencies above the cutoff.



Ideal filter Practical filter

The cut-off frequency ω_c of a passive filter is the frequency at which |H(jω)| is reduced to $\frac{1}{\sqrt{2}}$ of its original value, which corresponds to a decrease of 3 decibels (dB) or approximately 70.7% of the input voltage.

Cut-off frequency of a low-pass filter: $|H(j\omega)|_{\omega=\omega_c} = \frac{|H(j\omega)|_{\omega\to 0}}{\sqrt{2}}$

The canonical form of a first-order low-pass filter transfer function is often written as

$$H(j\omega) = \frac{H_0}{1 + j\frac{\omega}{\omega_c}}$$

Where H_0 is the filter gain at ω=0 and ω_c is the cut-off frequency.

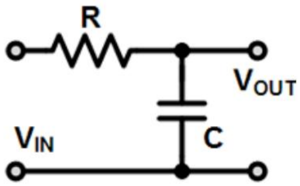
Key characteristics

- **Passband:** The range of frequencies from 0 Hz up to the cutoff frequency ω_c ($\omega < \omega_c$) that the filter passes with little to no attenuation.
- **Cutoff frequency ω_c:** The frequency at which the filter's gain drops to -3 dB (approximately half the power) below its passband gain.
- **Stopband:** The range of frequencies above the cutoff frequency where the signal is significantly attenuated ($\omega > \omega_c$).

- **Roll-off:** The rate at which the filter's gain decreases in the stopband. A simple first-order RC low-pass filter has a roll-off of -20 dB per decade (or -6 dB per octave).
- **Phase response:** As frequency increases from zero, the phase shift between the input and output gradually changes from 0° to a maximum of -90° at infinite frequency. It reaches -45° at the cutoff frequency (fcf sub cfc).

Example 2.3

Find the cut-off frequency of the following passive LPF.
R=100 Ω, C=10 nF.



- At **low frequencies**, the capacitor's reactance is very high, and it acts like an open circuit. This allows the low-frequency signal to pass almost completely to the output.

$$Z_C = \frac{1}{j\omega C} \Rightarrow |Z_C|_{\omega\to 0} \to \infty \text{ (open circuit).}$$

- At **high frequencies**, the capacitor's reactance is very low, causing it to act more like a short circuit. The high-frequency signal is shunted to the ground through the capacitor, leaving a negligible output.

$$Z_C = \frac{1}{j\omega C} \Rightarrow |Z_C|_{\omega\to\infty} \to 0 \text{ (short circuit).}$$

Transfer function can be found as follows

$$V_{OUT} = \frac{Z_C}{Z_C + R} V_{IN}$$

$$\Rightarrow H(j\omega) = \frac{V_{OUT}}{V_{IN}} = \frac{Z_C}{Z_C+R}\ \frac{Z_C}{Z_C+R} \Rightarrow H(j\omega) = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C}+R} \Rightarrow \boldsymbol{H(j\omega) = \frac{1}{1+j\omega RC}}$$

Cut-off frequency is determined as follows

$$|H(j\omega)|_{\omega=\omega_c} = \frac{|H(j\omega)|_{\omega\to 0}}{\sqrt{2}}$$

$$|H(j\omega)|_{\omega_c} = \frac{1}{\sqrt{1+(\omega_c RC)^2}} \quad \text{and} \quad |H(j\omega)|_{\omega\to 0} = \left.\frac{1}{\sqrt{1+(\omega_c RC)^2}}\right|_{\omega_c\to 0} = 1$$

Thus

$$\frac{1}{\sqrt{1+(\omega_c RC)^2}} = \frac{1}{\sqrt{2}} \Rightarrow \omega_c = \frac{1}{RC} = 10^6\ rad/s \Rightarrow f_c = \frac{1}{2\pi RC} = 159.15\ kHz$$

Plotting the transfer function $H(j\omega)$

```
clear;clc;
R=1e2; C=10e-9;
w0=1/(R*C);
w=logspace(1,10,100);
H=1./complex(1,w*R*C);
abs_H=abs(H);
angle_H=angle(H);
figure(1);
subplot(2,2,1);
semilogx(w,abs_H,'LineWidth',2); grid on;
xlabel('Frequency(rad/s)'); ylabel('|H(j\omega)|');
subplot(2,2,3);
semilogx(w,20*log10(abs_H),'LineWidth',2); grid on;
xlabel('Frequency(rad/s)'); ylabel('HdB');
subplot(2,2,4);
semilogx(w,angle_H*360/(2*pi),'LineWidth',2);
xlabel('Frequency(rad/s)'); ylabel('Phase(deg)');
%Using bode function
figure(2);
s = tf('s');
G = 1/(R*C*s + 1);
bode(G,w);
```

Bode plot:

The Bode Plot is the frequency response plot of linear systems represented in the form of logarithmic plots. In bode plot the horizontal axis represents frequency on a logarithmic scale and the vertical axis represents either the amplitude in decibels or the phase in rad (or deg) of the frequency response function.

- **Bode Magnitude Plot**

  $H_{dB} = 20\, log(|H(j\omega)|)$

  In a Bode plot, HdB which is the magnitude of a system's response in decibels, is plotted against frequency on a logarithmic scale.

- **Bode Phase Plot**

  Phase = $arg\big(H(j\omega)\big)$

  Bode plot shows phase in degrees or radians as a function of frequency on a logarithmic scale.

- The magnitude in decibels:

$$H_{dB}(\omega) = 20\,Log(|H(\omega)|) = 20\,Log\left(\frac{H_0}{\sqrt{1+\left(\frac{\omega}{\omega_c}\right)^2}}\right) = -10\,Log\left(1+\left(\frac{\omega}{\omega_c}\right)^2\right)$$

$$H_{dB}(\omega) = \begin{cases} 0 & \omega \ll \omega_c \\ -3 & \omega = \omega_c \\ -\infty & \omega \to \infty \end{cases}$$

The roll-off slope of a filter is the rate at which the magnitude of the frequency response decreases after the cutoff frequency. This is typically measured in **decibels per octave (dB/oct)** or **decibels per decade (dB/decade)**.

To find the roll-off slope in dB/decade (a tenfold increase in frequency), you would take the difference in decibel gain between two frequencies where one is ten times the other:

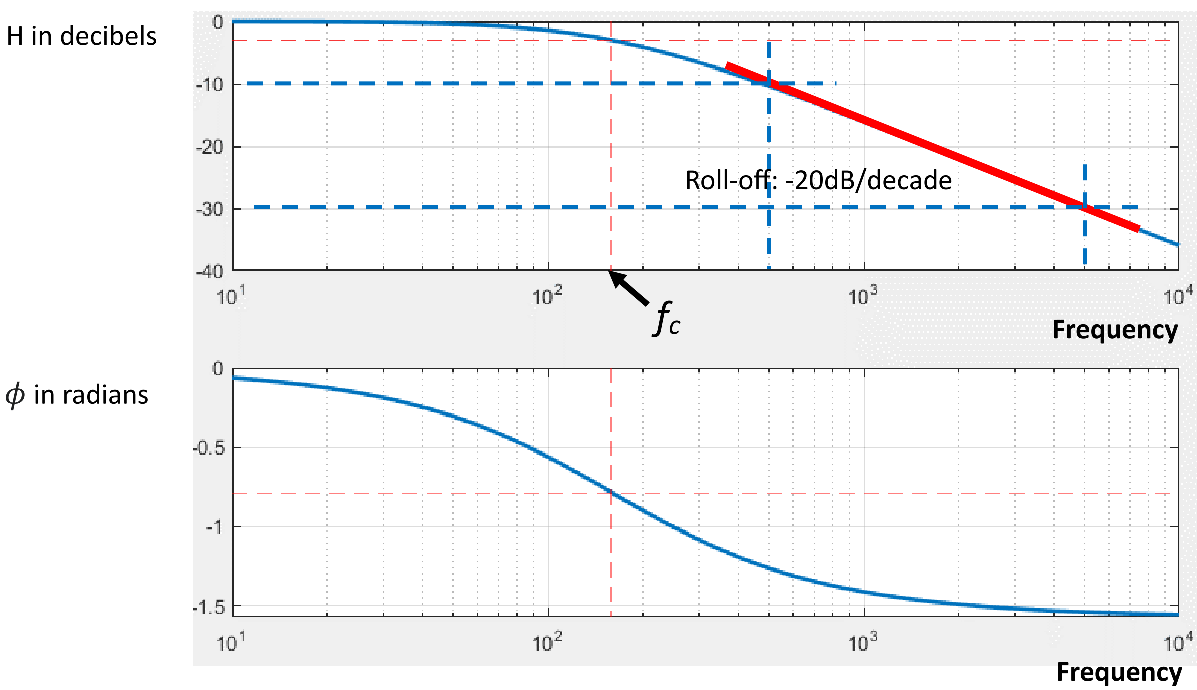
Roll-off slope = $H_{dB}(10\omega) - H_{dB}(\omega)$ where $\omega \gg \omega_c$.

$H_{dB}(\omega_1 \gg \omega_c) \approx -20\,Log\left(\frac{\omega_1}{\omega_c}\right)$

$H_{dB}(\omega_2 = 10\,\omega_1 \gg \omega_c) \approx -20\,Log\left(\frac{10\,\omega_1}{\omega_c}\right)$

Slope = $H_{dB}(\omega_2 = 10\,\omega_1) - H_{dB}(\omega_1) = -20\,dB/decade$

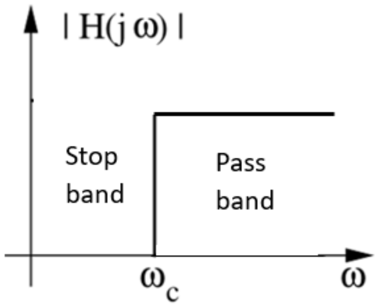
- The phase in radians : $\phi(\omega) = arctan(H_0) - arctan\left(\frac{\omega}{\omega_c}\right) = -arctan\left(\frac{\omega}{\omega_c}\right)$



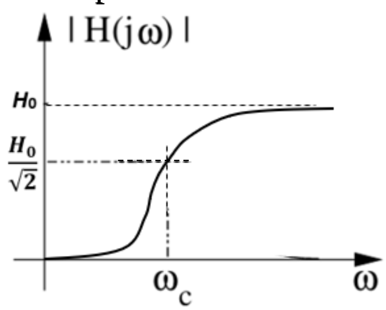
2-3-2- High-pass filter

A passive High-pass filter (HPF) is a two-port network composed of passive components, such as resistors (R), capacitors (C), and/or inductors (L), that works without an external power source.

- It allows signals with frequencies above a specified "cutoff frequency" ω_c to pass through with minimal attenuation.
- It blocks or attenuates signals with frequencies below the cutoff.



Ideal filter



Practical filter

The cut-off frequency ω_c of a passive filter is the frequency at which $|H(j\omega)|$ is reduced to $\frac{1}{\sqrt{2}}$ of its original value, which corresponds to a decrease of 3 decibels (dB) or approximately 70.7% of the input voltage.

Cut-off frequency of a high-pass filter: $|H(j\omega)|_{\omega=\omega_c} = \frac{|H(j\omega)|_{\omega\to\infty}}{\sqrt{2}}$

The canonical form of a first-order high-pass filter transfer function is often written as

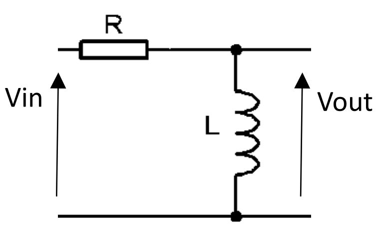
$$H(\omega) = H_0 \frac{j\omega}{\omega_c + j\omega}$$

Where H_0 is the filter gain at $\omega\to\infty$ and ω_c is the cut-off or corner frequency.

Example 2.4

Find the cut-off frequency of the following passive HPF.

R=100 Ω, L=10 μF.



Transfer function can be found as follows

$$V_{OUT} = \frac{Z_L}{Z_L + R} V_{IN}$$

⇨ $H(j\omega) = \frac{V_{OUT}}{V_{IN}} = \frac{Z_L}{Z_L+R}$ => $H(j\omega) = \frac{j\omega L}{j\omega L+R}$ => $\boldsymbol{H(j\omega) = \frac{j\omega}{j\omega+\frac{R}{L}}}$

Cut-off frequency is determined as follows

$$|H(j\omega)|_{\omega=\omega_c} = \frac{|H(j\omega)|_{\omega\to\infty}}{\sqrt{2}}$$

$$|\boldsymbol{H}(\boldsymbol{j\omega})|_{\omega_c} = \frac{\omega_c}{\sqrt{\omega_c^2+\left(\frac{R}{L}\right)^2}} \quad \text{and} \quad |\boldsymbol{H}(\boldsymbol{j\omega})|_{\omega\to\infty} = \left.\frac{\omega_c}{\sqrt{\omega_c^2+\left(\frac{R}{L}\right)^2}}\right|_{\omega_c\to\infty} = \mathbf{1}$$

Thus

$$\frac{\omega_c}{\sqrt{\omega_c^2+\left(\frac{R}{L}\right)^2}} = \frac{1}{\sqrt{2}} \Rightarrow \omega_c = \frac{R}{L} = 10^7\, rad/s \Rightarrow f_c = \frac{1}{2\pi RC} = \mathbf{1}.59\, MHz$$

2-3-3- Band-pass filter

The canonical form of a second-order band-pass filter is a transfer function of the form

$$H(j\omega) = H_0 \frac{j\omega}{(j\omega)^2 + \frac{\omega_0}{Q} j\omega + \omega_0^2}$$

where $H_0 \frac{Q}{\omega_0}$ is the gain at $\omega = \omega_0$, ω_0 is the center (or resonant) angular frequency, and Q is the quality factor.