Chapter IV

Hooke's law (Stress – Strain relation)

V-1- behavior law

The behavior law (constitutive law) describes how a material resists stress. This law depends on the point within the material considered, the direction, the temperature, and the time. It expresses the relationship between the stress tensor and the strain tensor.

$$\sigma_{ij} = \sigma_{ij}\,(\varepsilon_{hk}, x, T, t)$$

V-2- Constitutive law of a linear elastic solid

For a body to exhibit linear elastic behavior, at every point in the material and for a given temperature, a stress state must cause only one deformation state, and the local constitutive law must be invertible.

(Linear elasticity = geometric linearity + physical linearity)

This law is written in an orthonormal coordinate system in the following form:

$\sigma_{ij} = C_{ijhk}\ \varepsilon_{hk}$	or	$\{\sigma\}=[C]\ \{\varepsilon\}$	[C] : Stiffness matrix
$\varepsilon_{ij} = S_{ijhk}\ \sigma_{hk}$	or	$\{\varepsilon\}=[S]\ \{\sigma\}$	[S] : Campliance matrix

where C_{ijhk} (or S_{ijhk}) is a fourth-order elasticity tensor whose components include all the material parameters necessary to characterize the material. Based on the symmetry of the stress and strain tensors, the elasticity tensor must have the following properties:

$$\begin{cases} \varepsilon_{ij} = \varepsilon_{ji} \\ \sigma_{ij} = \sigma_{ji} \end{cases} \quad ===> \quad \begin{cases} C_{ijhk} = C_{jihk} \\ C_{ijhk} = C_{ijkh} \end{cases}$$

In general, the fourth-order tensor C_{ijhk} has 81 components. However, these relations reduce the number of independent components to 36.

$$\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112}\\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212}\\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312}\\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312}\\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312}\\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212}\end{bmatrix} \begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{23}\\ \varepsilon_{13}\\ \varepsilon_{12}\end{Bmatrix}$$

Taking into account energy considerations related to strain energy, the stiffness tensor C (or S) is symmetric. Therefore, in this case, the stiffness tensor requires only twenty-one (21) independent constants.

$$C_{ijhk} = C_{hkij} \quad ====> \begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112}\\ & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212}\\ & & C_{3333} & C_{3323} & C_{3313} & C_{3312}\\ & & & C_{2323} & C_{2313} & C_{2312}\\ & Sym & & & C_{1313} & C_{1312}\\ & & & & & C_{1212}\end{bmatrix} \begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{23}\\ \varepsilon_{13}\\ \varepsilon_{12}\end{Bmatrix}$$

A medium that requires 21 independent components is called a *linear anisotropic elastic* medium.
Remark : The *strain energy* is given by :

$$W = \frac{1}{2}\,\mathrm{tr}(\Sigma\cdot\mathscr{E}) = \frac{1}{2}\,\sigma_{ij}\,\varepsilon_{ij} = \frac{1}{2}\,C_{ijhk}\,\varepsilon_{hk}\,\varepsilon_{ij}$$

V-3- Isotropic elasticity

	σ_{11} (only)	σ_{22} (only)	σ_{33} (only)	σ_{23} (only)	σ_{13} (only)	σ_{12} (only)
ε_{11}	$\frac{1}{E}\sigma_{11}$	$-\frac{\upsilon}{E}\sigma_{22}$	$-\frac{\upsilon}{E}\sigma_{33}$			
ε_{22}	$-\frac{\upsilon}{E}\sigma_{11}$	$\frac{1}{E}\sigma_{22}$	$-\frac{\upsilon}{E}\sigma_{33}$			
ε_{33}	$-\frac{\upsilon}{E}\sigma_{11}$	$-\frac{\upsilon}{E}\sigma_{22}$	$\frac{1}{E}\sigma_{33}$			
ε_{23}				$\frac{1+\upsilon}{E}\sigma_{23}$		
ε_{13}					$\frac{1+\upsilon}{E}\sigma_{13}$	
ε_{12}						$\frac{1+\upsilon}{E}\sigma_{12}$

A material is said to be isotropic if every axis is an axis of symmetry. In this case, it requires two (2) independent components: E (modulus of elasticity or Young's modulus)

ν (Poisson's ratio)

The Generalized Hooke's Law

$\mathscr{E} = \frac{1+\nu}{E}\, \Sigma - \frac{\nu}{E}$ s I or $\varepsilon_{ij} = \frac{1+\nu}{E}\, \sigma_{ij} - \frac{\nu}{E}\, \sigma_{kk}\, \delta_{ij}$ s = tr(Σ)

$\Sigma = 2\mu\, \mathscr{E} + \lambda$ e I or $\sigma_{ij} = 2\mu\, \varepsilon_{ij} + \lambda\, \varepsilon_{kk}\, \delta_{ij}$ e = tr($\mathscr{E}$)

$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$ $\mu = \frac{E}{2(1+\nu)} = G$ [(λ , μ) : Lame´s coefficients]

[G : shear modulus]

In this case, the elasticity tensors are written as follows:

$$C = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & O & 0 & 2\mu \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} \frac{1}{E} & -\frac{\upsilon}{E} & -\frac{\upsilon}{E} & 0 & 0 & 0 \\ -\frac{\upsilon}{E} & \frac{1}{E} & -\frac{\upsilon}{E} & 0 & 0 & 0 \\ -\frac{\upsilon}{E} & -\frac{\upsilon}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1+\nu}{E} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1+\nu}{E} & 0 \\ 0 & 0 & 0 & O & 0 & \frac{1+\nu}{E} \end{bmatrix}$$

V-4- Thermoelastic Constitutive Relation

It is well known that a temperature change in an unrestrained elastic solid produces deformation. Thus, a general strain field results from both mechanical and thermal effects. Within the context of linear small deformation theory, the total strain can be decomposed into the sum of mechanical and thermal components as.

$$\varepsilon_{ij} = \varepsilon_{ij}^{(m)} + \varepsilon_{ij}^{(th)}$$

If T_o is taken as the reference temperature and T as an arbitrary temperature, the thermal strains in an unrestrained solid can be written in the linear constitutive form

$$\varepsilon_{ij}^{(th)} = \alpha(T - T_0)\delta_{ij} = \alpha(\Delta T)\delta_{ij}$$

where α is a material constant called the coefficient of thermal expansion. Notice that for isotropic materials, no shear strains are created by temperature change. this result can be combined with the mechanical relation to give

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\,\delta_{ij} + \alpha(T - T_0)\delta_{ij}$$

$$\sigma_{ij} = 2\mu\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\,\delta_{ij} - \alpha(3\lambda + 2\mu)(T - T_0)\delta_{ij}$$

Table provides typical values of this constant for some common materials

	E (GPa)	ν	α(10^{-6}/°C)
Steel	207	0.29	13.5
Copper	89.6	0.34	18
Aluminum	68.9	0.34	25.5
Glass	68.9	0.25	8.8
Nylon	28.3	0.40	102
Rubber	0.0019	0.499	200