

Graph Theory

2nd year computer science L2

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Chapitre 6

The Maximum Flow Problem

The Maximum Flows

What is a flow?

A flow in a graph models the circulation of a product on the network represented by that graph.

Examples of flow modeling:

1. **Transportation flows on roads:** $C(u, v)$ is the maximum number of cars that can travel from u to v in one hour.

For instance, if we consider a graph representing roads (with vertices as intersections and edges as road segments), we associate a capacity with each edge (the maximum number of cars per hour, for example), and the question is to determine the maximum possible flow of cars between two points in the graph.

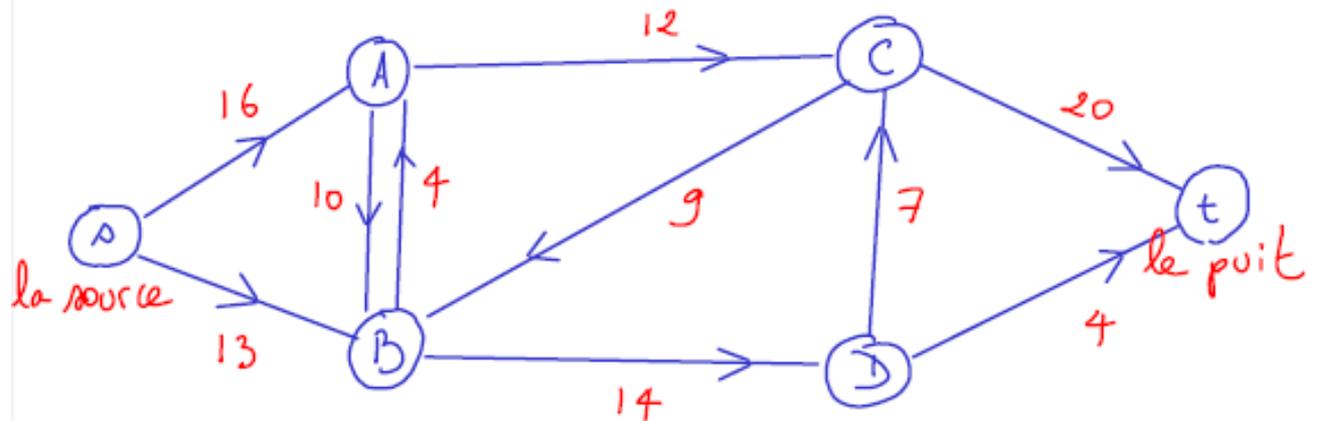
2. **Data transport:** ADSL line (bandwidth), truck capacity, bus seating capacity.
3. **Flow of** water, gas, oil.

The Maximum Flows

- **Definition: (network)**

A graph is called a network if it is a directed weighted graph, denoted as $N=(V,E,C)$:

- **V**: the set of vertices. Among these vertices, two special vertices are:
 - **the source**, denoted as **s**, which is a vertex without a predecessor (zero in-degree).
 - **the sink**, denoted as **t**, which is without a successor (zero out-degree).
- **E**: the set of arcs in the network.
- **C**: the set of capacities, for each arc $(i,j) \in E$, there exists a non-negative capacity $C(i,j)$.



The Maximum Flows

Definition: (flow through a network)

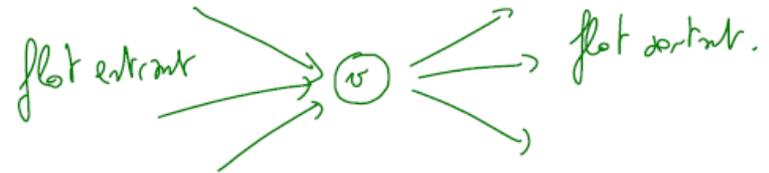
A flow of rate F in a network $R=(V,E,C)$ is a mapping Φ from E to \mathbb{R} satisfying:

- **Capacity constraints** on each arc:

$$\forall (i,j) \in E, \quad 0 \leq \Phi(i,j) \leq C(i,j)$$

- The **law of conservation** (Kirchhoff's law: what **goes in** equals what **goes out**) at each node (except the source and the sink):

$$\forall i \neq s, t, \quad \sum_{j \in \Gamma^+(i)} \Phi_{i,j} = \sum_{j \in \Gamma^-(i)} \Phi_{j,i}$$



- The flow F leaving the **source** s enters the **sink** t

$$F = \sum_{j \in \Gamma^+(s)} \Phi_{s,j} = \sum_{j \in \Gamma^-(t)} \Phi_{j,t}$$

The Maximum Flows

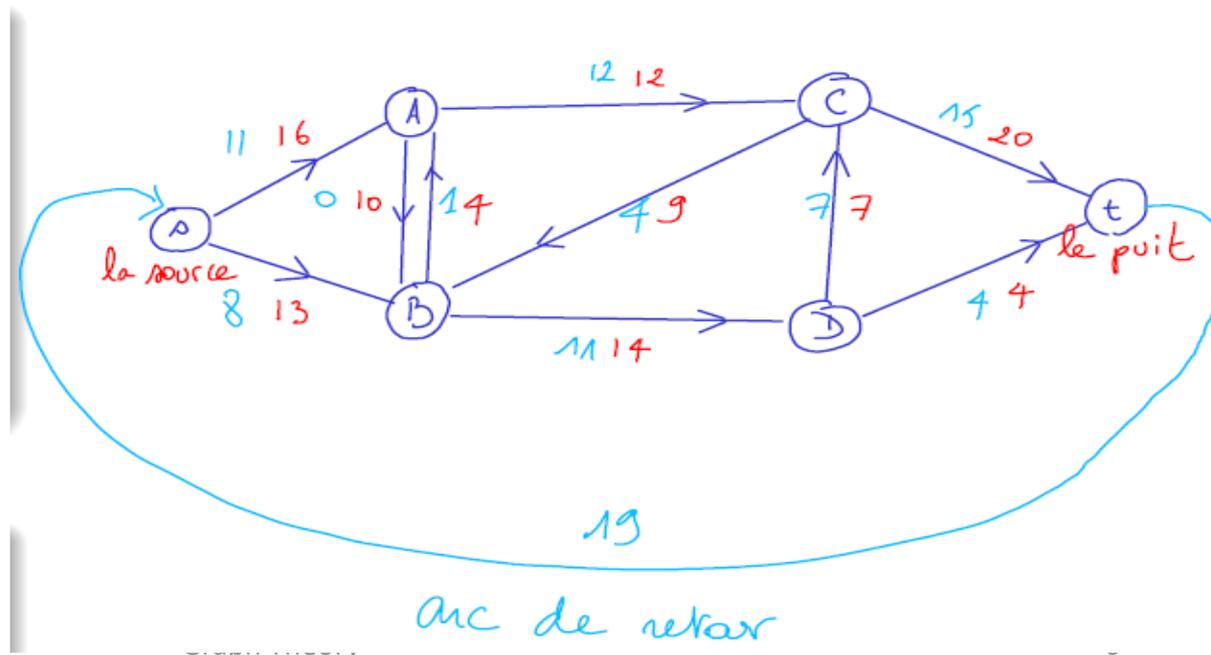
Definition: (Feasible Flow)

If, for every arc, the flow value is less than or equal to the capacity of the arc, then the flow is said to **be feasible**.

Definition: (Flow Value)

A fictitious **return arc** is added from **t** to **s**. The flow value is then defined as the flow passing through this **fictitious arc.(fictif)**

- Capacity in **red**
- Flow value in **blue**
- The flow has a rate of **19**
- For each vertex (except s and t), **flow conservation** is satisfied



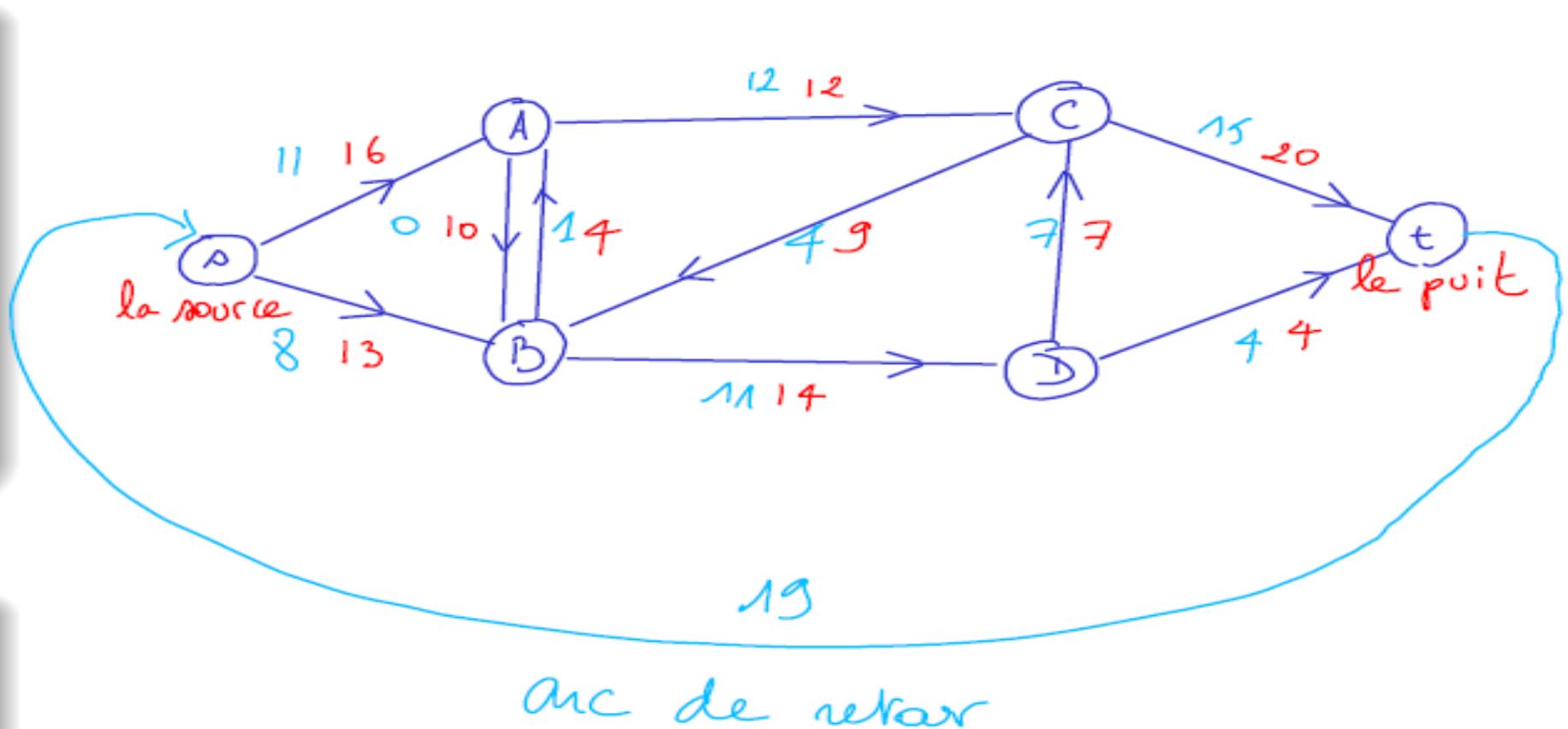
Example: Vertex B

Inflow: $(8 + 4 + 0) = 12$

Outflow: $(11 + 1) = 12$

The Maximum Flows

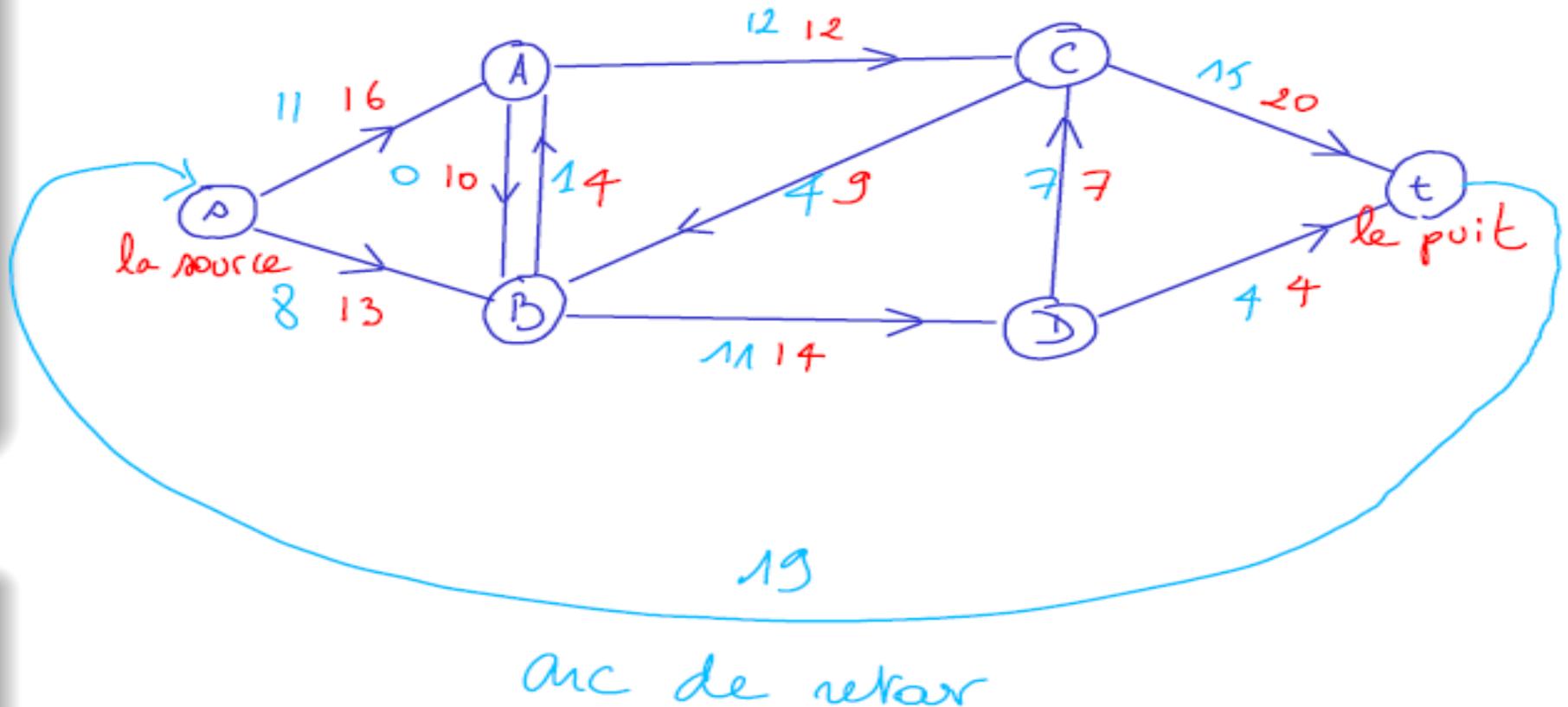
- All the flow leaving the source s reaches the sink $t = 19$
- Outflow from source $s = 11 + 8 = 19$, Inflow to sink $t = 15 + 4 = 19$
- All capacities are respected: blue value is less than or equal to red



The Maximum Flows

Saturated Arc:

- An arc is said to be **saturated** if the flow on that arc is equal to the capacity of the arc.
- Arcs (A, C), (D, C), (D, t) are saturated. The other arcs are unsaturated



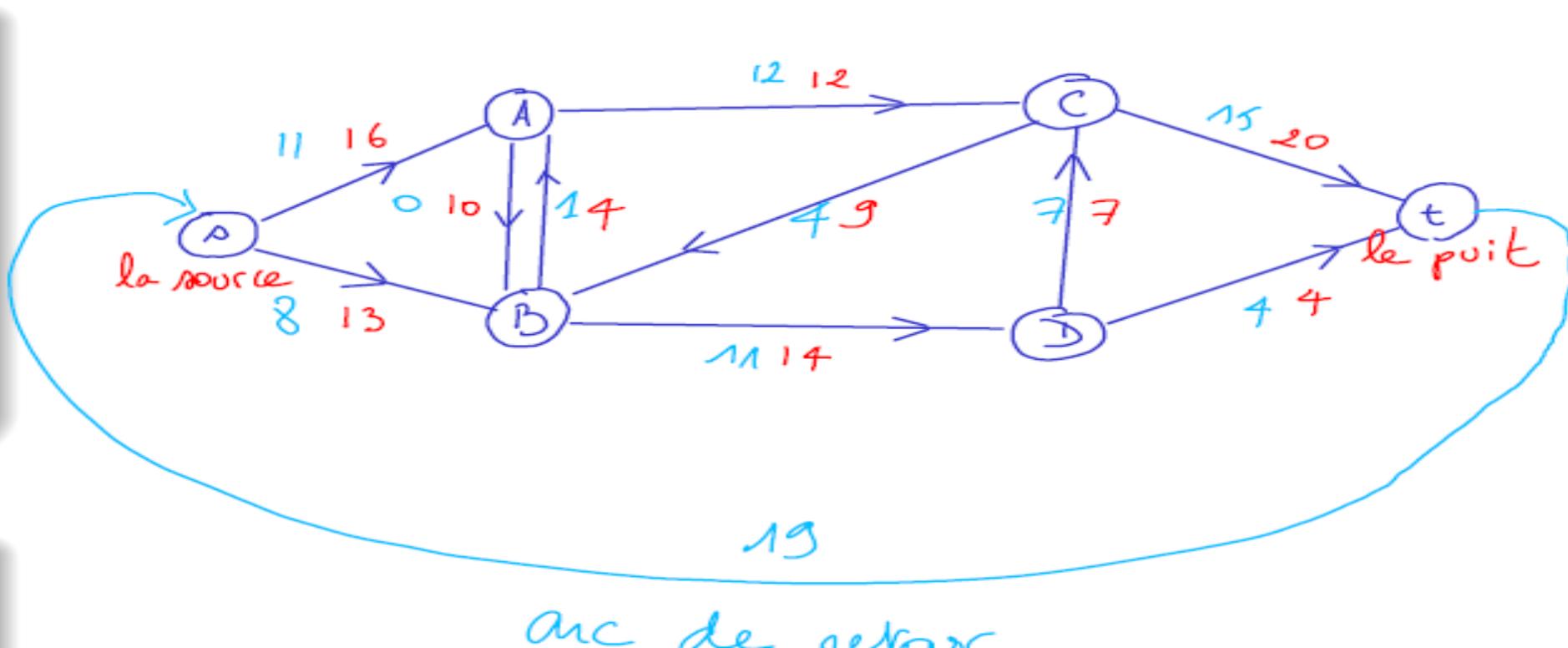
The Maximum Flows

Complete Flow:

A flow is complete if, for every path from the source to the sink, there is at least one saturated arc.

Examples:

- First path: s, A, C, t arc (A, C) is saturated
- Second path: s, A, B, D, t arc (D, t) is saturated



The Maximum Flows

The Maximum Flow Problem : involves finding the maximum quantity of flow to be routed from the source to the sink, taking into account the capacities and the available flow quantity.

Problem:

- **Given:** a network N
- **Question:** find a maximal feasible flow (with the maximum value)
- **Solution:** **Ford-Fulkerson Algorithm (1956)**

Algorithm:

Initialize the flow f to 0 for all arcs in the network

While N has **an augmenting path** C **do**

 Calculate the possible flow increase ϵ along C

 increase the flow by ϵ along C

End while

Return the flow f

The Maximum Flows

- The principle of the Ford-Fulkerson algorithm is to pass **a feasible** (compatible) flow through the network, initially the most obvious being the **zero flow**, and then improve it until **a complete flow** is achieved (**all paths are saturated**).

An augmenting path is a path whose flow can be increased, and it is a path where the arcs in the forward direction have not reached their limit, and the arcs in the reverse direction have a non-zero flow.

In other words, a path C is said to be **augmenting** if:

For every direct arc u in C , $u \in C^+$: $f(u) < C(u)$

For every indirect arc u in C , $u \in C^-$: $f(u) > 0$

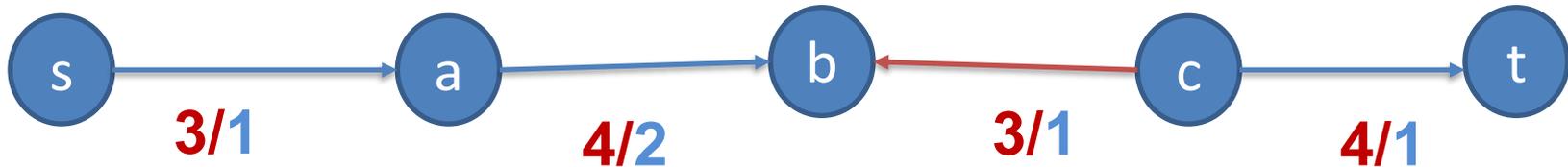
The flow on this path C can be **increased** by the following value:

$$\varepsilon = \text{minimum} \{ (C(u) - f(u) / u \in C^+), (f(u) / u \in C^-) \}$$

To improve the flow, ε is **added** to the flows of the arcs in C^+ and **subtracted** from the flows of the arcs in C^-

The Maximum Flows

Example: A path C connecting vertices s and t taken from a network whose flow can be increased.



Arcs in $C^+ = \{(s, a), (a, b), (c, t)\}$ satisfy $f(u) < C(u)$

Arcs in $C^- = \{(c, b)\}$ satisfy $f(u) > 0$

→ **Path C is augmenting.**

The flow on this path can be increased by the following value:

$$\varepsilon = \text{minimum} \{(C(u) - f(u) / u \in C^+), (f(u) / u \in C^-)\}$$

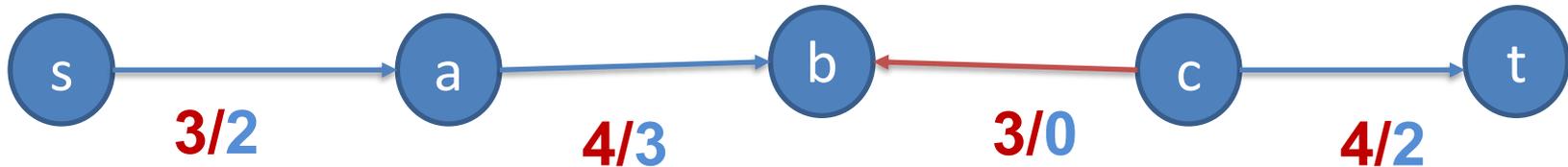
$$= \text{minimum} \{(s, a), (a, b), (c, t), (c, b)\} = \text{minimum} (3-1, 4-2, 4-1, 1) = 1$$

$$= \text{minimum} (3-1, 4-2, 4-1, 1) = \min(2, 2, 3, 1) = 1$$

→ We increase the flow on the path by the value $\varepsilon=1$

The Maximum Flows

Example: A path C connecting vertices s and t taken from a network whose flow can be increased.



→ We increase the flow on the path by the value 1.

Increase the flow from s to a by 1.

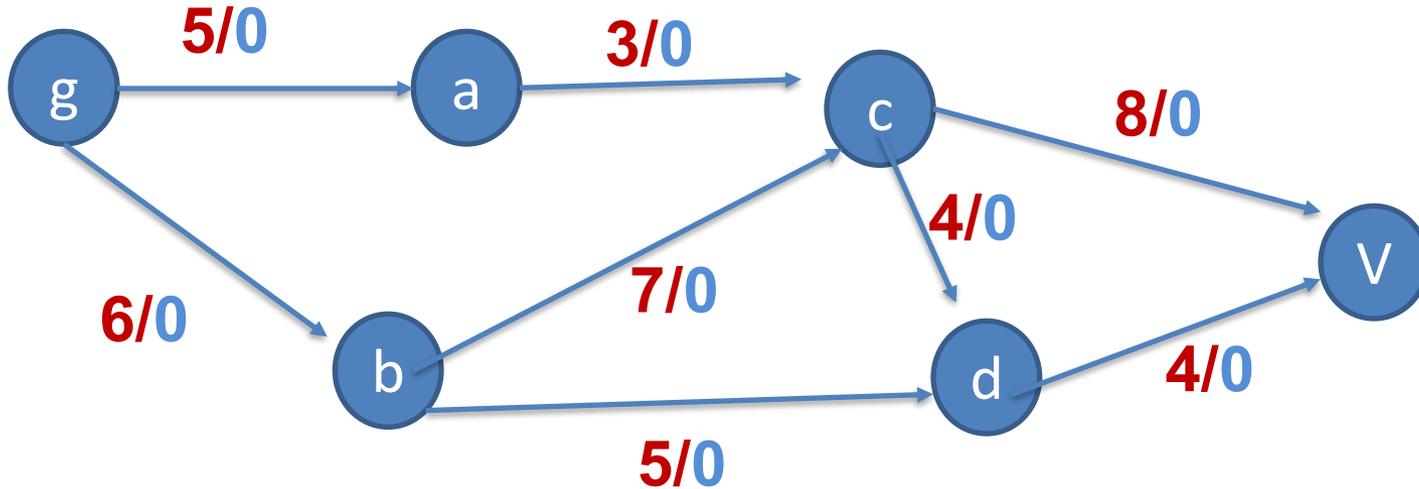
Increase the flow from a to b by 1.

Decrease the flow from c to b by 1.

Increase the flow from c to t by 1.

The Maximum Flows

Application: Flow Maximization A gas plant supplies a city V through the following distribution network. The values represent transport capacities.



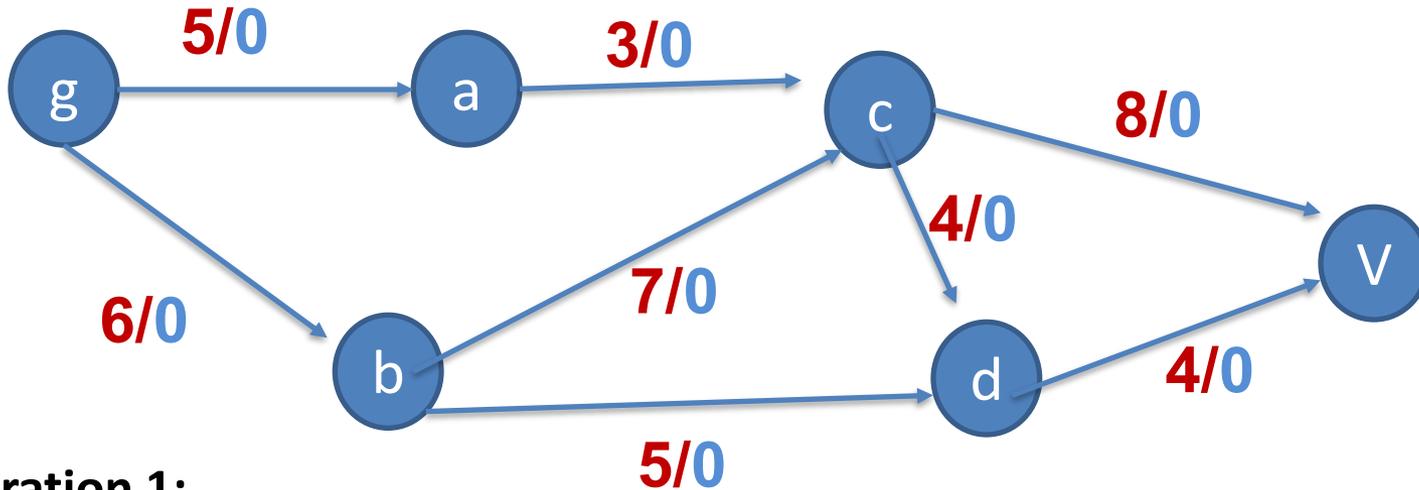
Initialization:

The flow of the arcs is initialized to 0.

We look for augmenting paths connecting vertex g and vertex V

The Maximum Flows

Application: Flow Maximization A gas plant supplies a city V through the following distribution network. The values represent transport capacities.



Iteration 1:

$C_1 = \{g, a, c, V\}$

Path C_1 is augmenting.

The flow on this path can be increased by the following value:

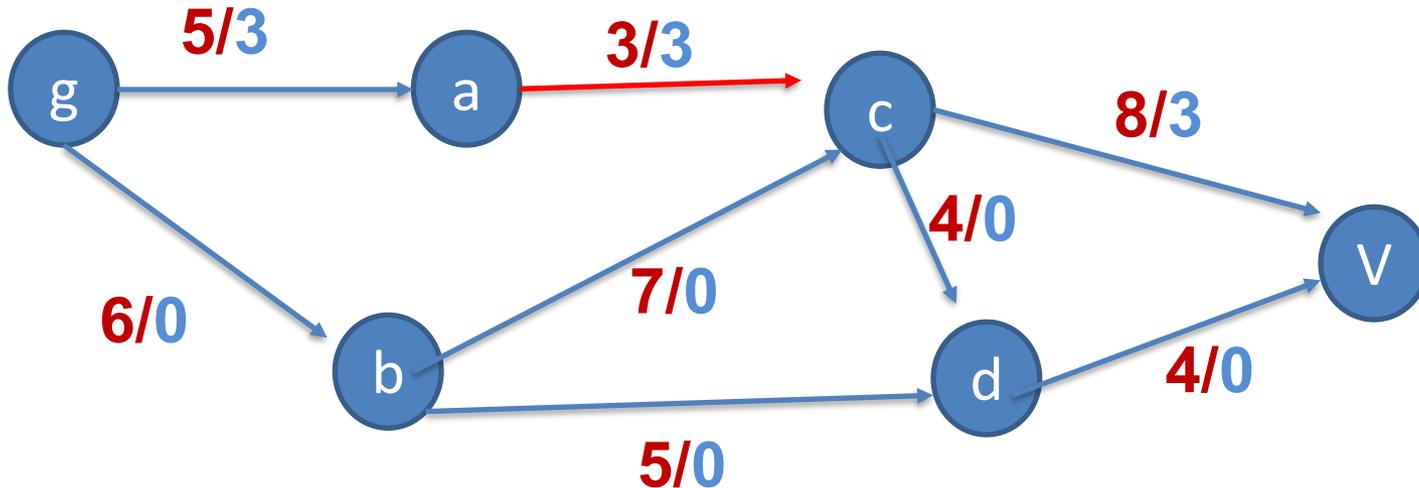
$$\epsilon_1 = \text{minimum} \{(C(u) - f(u) / u \in C^+), (f(u) / u \in C^-)\}$$

$$= \text{minimum} \{(g, a), (a, c), (c, V)\} = \text{minimum} (5-0, 3-0, 8-0) = 3$$

→ We increase the flow on the path by the value 3

The Maximum Flows

Application: Flow Maximization A gas plant supplies a city V through the following distribution network. The values represent transport capacities.



The arc (a, c) becomes saturated.

Iteration 2:

C2 = {g, b, c, d, V} The path C2 is augmenting.

The flow on this path can be increased by the following value:

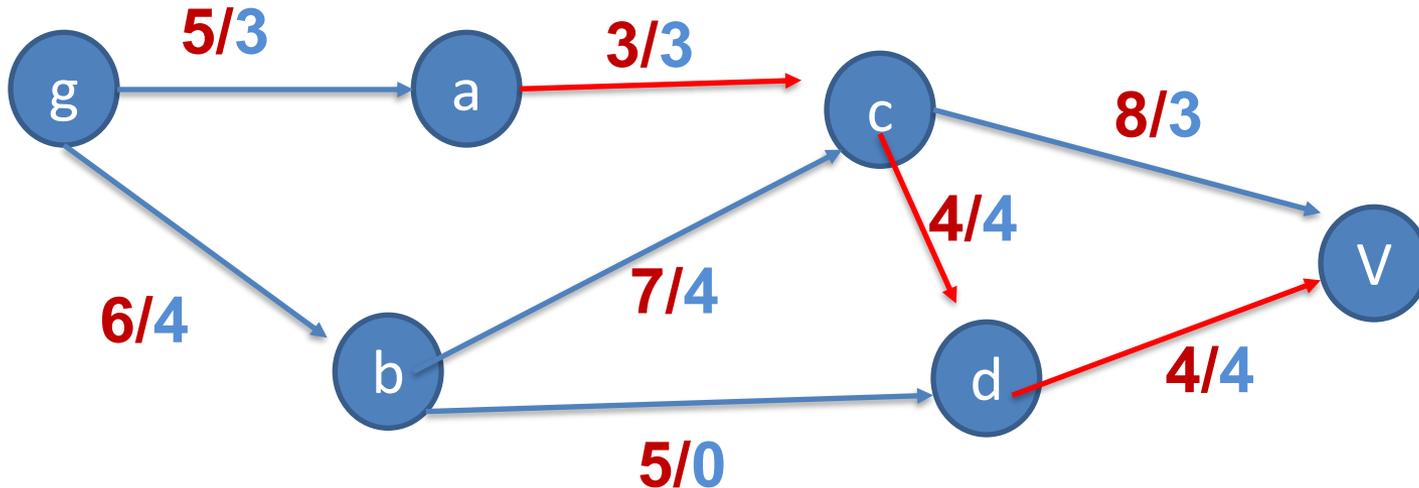
$$\varepsilon_2 = \text{minimum} \{(C(u) - f(u) / u \in C^+), (f(u) / u \in C^-)\}$$

$$= \text{minimum} \{(g, b), (b, c), (c, d), (c, V)\} = \text{minimum} (6-0, 7-0, 4-0, 4-0) = 4$$

→ We increase the flow on the path by the value 4

The Maximum Flows

Application: Flow Maximization A gas plant supplies a city V through the following distribution network. The values represent transport capacities.



The arcs (c, d) and (d, V) become saturated.

Iteration 3:

C3 = {g, b, d, c, V} The path C3 is augmenting.

$$\varepsilon_3 = \text{minimum} \{(C(u) - f(u) / u \in C^+), (f(u) / u \in C^-)\}$$

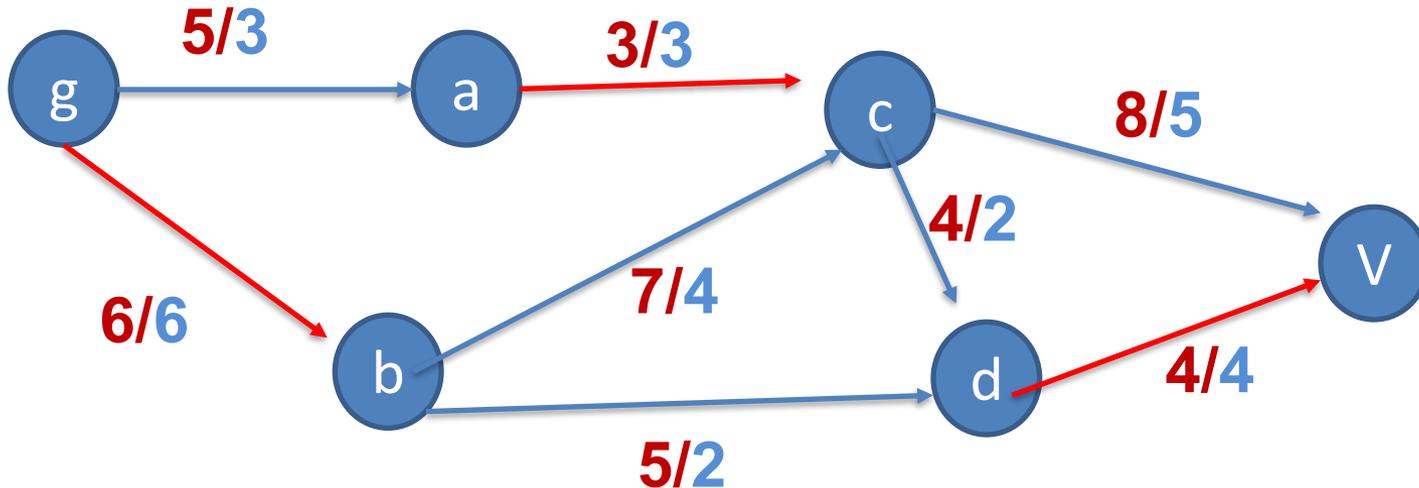
$$= \text{minimum} \{(g, b), (b, d), (c, V), (d, c)\} = \text{minimum} (6-4, 5-0, 8-3, 4) = (2, 5, 5, 4) = 2$$

→ We increase the flow on the path by the value 2

We can chose an other path C=(g,B,C,V)

The Maximum Flows

Application: Flow Maximization A gas plant supplies a city V through the following distribution network. The values represent transport capacities.



The arc (g, b) becomes saturated.

Iteration 3:

All paths are saturated → Complete Flow (maximum flow)

The flow value = $f(g, a) + f(g, b) = 3 + 6 = 9$ or $f(c, V) + f(d, V) = 5 + 4 = 9$

The Maximum Flows

Remarks:

To speed up the search for the maximum flow, we start with a positive, compatible, and feasible flow that adheres to the flow rules

Example 2

The following graph represents the internal layout of a parking , where vertex S represents the entrance of the parking and vertex P represents the exit of the parking.

All vehicles are heading towards the exit.

The table below provides the lane capacities (in vehicles per hour) and the observed flows of cars one evening between 4 pm and 5 pm.

Questions:

1- Is this flow compatible with the capacity constraints, and does it obey Kirchhoff's law?

2- Is this flow complete?

3- If the flow from S to A increases by 1, will Kirchhoff's law still be satisfied? Discuss,

Exemple 2

lane	Capacity	Real Flot
S-A	30	11
S-B	25	25
S-C	20	15
A-E	15	11
B-E	23	15
B-D	10	10
C-D	15	15
D-E	14	14
D-P	30	11
E-P	40	40

Exemple 2

1- The flow is compatible (with capacities).

All vertices satisfy Kirchhoff's law.

2- For every path from S to P, there is at least one saturated arc, so all paths are saturated, hence the flow is complete.

3- If we increase the flow on arc (S, A) by 1, Kirchhoff's law will not be satisfied (we are talking about a bottleneck at E).

Let's look for non-saturated paths.

We find the following augmenting path: S-A-E-D-P

Epsilon = 4

The new flow is now at its maximum.

Example 3

How to modelise and calculate the maximum flow for a graph with multiple source flow ?

Example 4

Flow in a Bipartite Network:

A bipartite network consists of two sets of vertices:

Left set $U = \{u_1, u_2, u_3\}$ Right set $V = \{v_1, v_2, v_3\}$

The edges connecting the vertices have the following capacities:

$u_1 \rightarrow v_1$: capacity = 10

$u_1 \rightarrow v_2$: capacity = 5

$u_2 \rightarrow v_2$: capacity = 10

$u_2 \rightarrow v_3$: capacity = 15

$u_3 \rightarrow v_1$: capacity = 5

$u_3 \rightarrow v_3$: capacity = 10

1) Transform this problem into a maximum flow problem by adding a source S and a sink T .

2) Calculate the maximum flow from the source S to the sink T .

Example 5

Capacities:

- $s \rightarrow A : 16$
- $s \rightarrow B : 13$
- $A \rightarrow B : 10$
- $B \rightarrow A : 4$
- $A \rightarrow C : 12$
- $B \rightarrow D : 14$
- $C \rightarrow B : 9$
- $C \rightarrow t : 20$
- $D \rightarrow C : 7$
- $D \rightarrow t : 4$

Apply an algorithm to calculate the maximum flow

Example 5

Show that if G is a weighted graph and e is an edge whose weight is less than of any other edge, then e must belong to every minimum spanning tree of G .

Graph Theory

END and good luck

Any questions please?

Overall examination on Monday, 15/12/2025

G7+G8+G9+G10+G11+G12 8h

G1+G2+G3+G4+G5+G6 8h45