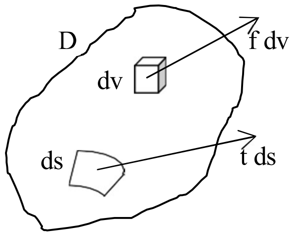
Chapter III

Stress Tensor

III-1 Types of external forces

Let D be a material domain of volume (V) and exterior surface (S).



The external forces applied to the domain are of two types:

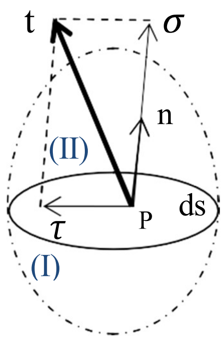
- Body forces acted on all particles of D.

On a volume dv, the resultant force is f·dv (f: [N/m³])

- Surface forces exerted on the particles constituting the boundary(S) of D.

On a surface ds, the resultant force is t·ds (t: [N/m²])

III-2- Stress Vector



Consider an elementary domain divided into two regions (I) and (II). The two regions are separated by a surface (ds).

The resultant contact forces of (II) on (I) are: $\vec{t}$·ds

($\vec{t}$) is the **stress vector** at point P relative to $\vec{n}$.

$$\vec{t} = \vec{\sigma} + \vec{\tau}$$

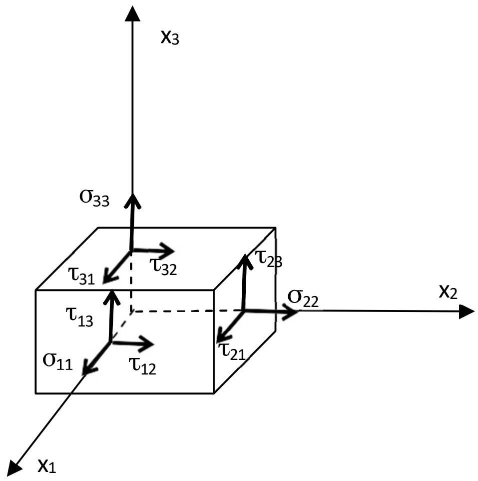
σ is the projection of t onto n, called the **normal stress**.

τ is the projection of t onto (ds), called the **shear stress**.

III-3- Stress Tensor

Cauchy's Theorem: At every point and at every instant, the dependence of the stress vector t on the normal n is linear. Therefore, there exists a second-order tensile field such that:

$$\vec{t} = \Sigma \vec{n} \qquad \text{ou} \quad t_i = \sigma_{ij}\, n_j$$



The stress tensor is defined by:

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{pmatrix}$$

σ_{ii} : normal stress

τ_{ij} : shear stress

Thus, the stress vector can therefore be decomposed into:

normal stress $\sigma = \vec{n}^t \, \Sigma \, \vec{n}$

shear stress $\tau = |(*\,\vec{n})\ \Sigma\ \vec{n})| = \sqrt{|\Sigma(\vec{n})|^2 - \sigma^2}$

III-4- Equilibrium equations

The equations of equilibrium express the vanishing of the sum of forces and the sum of moments about a point.

$$\iint_s \vec{t}\ ds + \iiint_v \vec{f}\ dv = \vec{0}$$

$$\iint_s \overrightarrow{OP} \wedge \vec{t}\ ds + \iiint_v \overrightarrow{OP} \wedge \vec{f}\ dv = \vec{0}$$

By projecting onto the axis e_i (i=1, 2, 3) we obtain :

$$\iint_s \sigma_{ij}\, n_j\ ds + \iiint_v f_i\ dv = 0$$

$$\iint_s (x_i\, \sigma_{jk} n_k - x_j\, \sigma_{ik} n_k)\ ds + \iiint_v (x_i\, f_j - x_j\, f_i)\ dv = 0$$

The divergence theorem allows the transformation of the double integral into a triple integral :

$$\iiint_v \left(\frac{\partial \sigma_{ij}}{\partial \sigma_j} + f_i\right) dv = 0$$

$$\iiint_v \left(\frac{\partial (x_i\, \sigma_{jk} - x_j\, \sigma_{ik})}{\partial \sigma_j} + x_i\, f_j - x_j\, f_i\right) dv = 0$$

The second equilibrium equation reflects the symmetry of the stress tensor : $\sigma_{ij} = \sigma_{ji}$

The stress tensor can then be written as:

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33} \end{pmatrix}$$

The first equilibrium equation reflects the local equilibrium of forces: $\frac{\partial \sigma_{ij}}{\partial \sigma_j} + f_i = 0$

or in vectorial form: $\vec{f} + \mathrm{div}(\Sigma) = \vec{0}$

These equations are written in:

Cartesian coordinates

$$f_1+\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}+\frac{\partial\sigma_{13}}{\partial x_3}=0$$
$$f_2+\frac{\partial\sigma_{12}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+\frac{\partial\sigma_{23}}{\partial x_3}=0$$
$$f_3+\frac{\partial\sigma_{13}}{\partial x_1}+\frac{\partial\sigma_{23}}{\partial x_2}+\frac{\partial\sigma_{33}}{\partial x_3}=0$$

cylindrical coordinates:

$$f_r+\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\left(\frac{\partial\sigma_{r\theta}}{\partial\theta}+\sigma_{rr}+\sigma_{\theta\theta}\right)+\frac{\partial\sigma_{r3}}{\partial x_3}=0$$
$$f_\theta+\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{1}{r}\left(\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+2\sigma_{r\theta}\right)+\frac{\partial\sigma_{\theta3}}{\partial x_3}=0$$
$$f_3+\frac{\partial\sigma_{r3}}{\partial r}+\frac{1}{r}\left(\frac{\partial\sigma_{\theta3}}{\partial\theta}+\sigma_{r3}\right)+\frac{\partial\sigma_{33}}{\partial x_3}=0$$

IV-5- Principal Stress and principal directions

In the principal basis frame $\{X_1, X_2, X_3\}$ the stress tensor is written:

$$\Sigma = \begin{pmatrix} \sigma_I & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{pmatrix}$$

To determine the principal elongations σ_I , σ_{II} et σ_{III}, the following equation must be solved:

$$\det(\Sigma - \lambda I) = 0$$

Expanding the determinant produces a cubic equation in terms of λ *(called characteristic equation)*:

$$det[\Sigma - \lambda I] = -\lambda^3 + s_1\lambda^2 - s_2\lambda + s_3 = 0$$

The roots of the characteristic equation determine the allowable values for λ $(\lambda_1, \lambda_2, \lambda_3)$ such as :

$\sigma_I=\lambda_1$ $\sigma_{II}=\lambda_2$ $\sigma_{III}=\lambda_3$

Where the scalars s_1, s_2, and s_3 are called the fundamental invariants of the tensor Σ such as :

$$s_1 = tr(\Sigma) = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_I + \sigma_{II} + \sigma_{III}$$
$$s_2 = \tfrac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}) = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{13} & \sigma_{33} \end{vmatrix} =$$
$$= \sigma_I\sigma_{II} + \sigma_{II}\sigma_{III} + \sigma_I\sigma_{III}$$
$$s_3 = \det(\Sigma) = det[\sigma_{ij}] = \sigma_I \cdot \sigma_{II} \cdot \sigma_{III}$$

To determine the principal directions X_1 , X_2 et X_3 the following vectorial equation must be solved:

$$(\Sigma - \sigma_i I)\,\vec{X}_i = \vec{0}$$

The unit vectors X_1, X_2 and X_3 of the principal basis verify the following vector relations:

$$\begin{cases} X_1 = X_2 \wedge X_3 \\ X_2 = X_3 \wedge X_1 \\ X_3 = X_1 \wedge X_2 \end{cases}$$

IV-6- Spherical and Deviatoric Stress

In particular applications it is convenient to decompose the stress tensor into two parts called spherical and deviatoric parts.

$\Sigma = \Sigma_s + \Sigma_d$ avec $\Sigma_s = \frac{s_1}{3} I$ $(s_1 = tr(\Sigma))$: the spherical part of Σ

$\Sigma_d = \Sigma - \frac{s_1}{3} I$: the deviatoric part of Σ

IV-7- Octahedral tension and shear

Octahedral tension and shear are defined for the direction

$$n_o = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \quad \text{such as} \quad \alpha^2 = \beta^2 = \gamma^2$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \qquad \alpha^2 = \beta^2 = \gamma^2 = 1/3 \qquad \Rightarrow \qquad n_o = \frac{\sqrt{3}}{3} \begin{pmatrix} \pm 1 \\ \pm 1 \\ \pm 1 \end{pmatrix}$$

Octahedral tension σ_o :

$$\sigma_o = \vec{n}_o^t \, \Sigma \, \vec{n}_o = \frac{1}{3}(\sigma_I + \sigma_{II} + \sigma_{III}) = \frac{1}{3} s_1$$

Octahedral shear τ_o :

$$\tau_o = \left|(*\vec{n}_o)\, \Sigma\, \vec{n}_o\right| = \sqrt{\left[\Sigma(\vec{n}_o)\right]^2 - \sigma_o^2} = \frac{1}{3}\sqrt{(\sigma_I - \sigma_{II})^2 + (\sigma_I - \sigma_{III})^2 + (\sigma_{II} - \sigma_{III})^2}$$

IV-8- Special states of stress

Spherical state of stress: $\sigma_I = \sigma_{II} = \sigma_{III} = s_1/3$

Uniaxial state of stress: $\sigma_{II} = \sigma_{III} = 0$ et $\sigma_I \neq 0$

Pure shear state: $\sigma_{ij} = 0$ et $\tau_{12} \neq 0$

Plane stress state:

$$\Sigma = \begin{pmatrix} \sigma_{11}(x_1, x_2) & \tau_{12}(x_1, x_2) & 0 \\ \tau_{12}(x_1, x_2) & \sigma_{22}(x_1, x_2) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$