Chapter 2

Real-valued Functions

Serie N° 2: On Numerical Functions

Exercise 2.1 (Domain Determination)

Find the domain of definition for the following function:

1. $f(x) = \frac{\sqrt{x-1}}{x^2-4}$
2. $g(x) = \sqrt{4-x^2} + \frac{1}{x-1}$
3. $h(x) = \ln(x+3) + \sqrt{9-x^2}$

Solution: Example 1

$$f(x) = \frac{\sqrt{x-1}}{x^2-4}$$

1. **Square root condition**: $x - 1 \geq 0 \Rightarrow x \geq 1$
2. **Denominator condition**: $x^2 - 4 \neq 0 \Rightarrow x \neq \pm 2$
3. **Combined conditions**: $x \geq 1$ and $x \neq 2$
4. **Final domain**: $[1, 2) \cup (2, +\infty)$

Solution: Example 2

$$g(x) = \sqrt{4-x^2} + \frac{1}{x-1}$$

1. **Square root condition**: $4 - x^2 \geq 0 \Rightarrow x^2 \leq 4 \Rightarrow -2 \leq x \leq 2$
2. **Denominator condition**: $x - 1 \neq 0 \Rightarrow x \neq 1$

3. **Combined conditions**: $-2 \leq x \leq 2$ and $x \neq 1$

4. **Final domain**: $[-2, 1) \cup (1, 2]$

Solution: Example 3

$$h(x) = \ln(x+3) + \sqrt{9 - x^2}$$

1. **Logarithm condition**: $x + 3 > 0 \Rightarrow x > -3$

2. **Square root condition**: $9 - x^2 \geq 0 \Rightarrow x^2 \leq 9 \Rightarrow -3 \leq x \leq 3$

3. **Combined conditions**: $x > -3$ and $-3 \leq x \leq 3 \Rightarrow -3 < x \leq 3$

4. **Final domain**: $(-3, 3]$

Exercise 2.2 (Parity Analysis)

Determine if the function is even, odd, or neither:

1. $f(x) = \frac{x^3 - x}{x^2 + 1}$

2. $g(x) = x^4 - 2x^2 + 5$

3. $h(x) = x^3 + x^2 + 1$

Solution : Example 1

$$f(x) = \frac{x^3 - x}{x^2 + 1}$$

$$f(-x) = \frac{(-x)^3 - (-x)}{(-x)^2 + 1} = \frac{-x^3 + x}{x^2 + 1}$$
$$= -\frac{x^3 - x}{x^2 + 1} = -f(x)$$

Since $f(-x) = -f(x)$, the function is **odd**.

Solution : Example 2

$$g(x) = x^4 - 2x^2 + 5$$

$$g(-x) = (-x)^4 - 2(-x)^2 + 5 = x^4 - 2x^2 + 5 = g(x)$$

Since $g(-x) = g(x)$, the function is **even**.

Solution : Example 3

$$h(x) = x^3 + x^2 + 1$$

$$h(-x) = (-x)^3 + (-x)^2 + 1 = -x^3 + x^2 + 1$$
$$h(-x) \neq h(x) \quad \text{and} \quad h(-x) \neq -h(x)$$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, the function is **neither even nor odd**.

Exercise 2.3 (Piecewise Function Analysis)

Given the piecewise functions, evaluate at specified points and analyze:

$$f(x) = \begin{cases} x^2 & \text{if} \quad x < 0 \\ 2x + 1 & \text{if} \quad x \geq 0 \end{cases}, \; g(x) = \begin{cases} |x| & \text{if} \quad x \leq 1 \\ \sqrt{x-1} & \text{if} \quad x > 1 \end{cases}, \; h(x) = \begin{cases} \sin x & \text{if} \quad x < 0 \\ \cos x & \text{if} \quad x \geq 0 \end{cases}$$

1. Find $f(-2)$, $f(0)$, $f(3)$ and sketch the graph.
2. Find $g(-1)$, $g(1)$, $g(5)$.
3. Find $h(-\pi/2)$, $h(0)$, $h(\pi)$.

Solution : Example 1

$$f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 2x + 1 & \text{if } x \geq 0 \end{cases}$$

Find $f(-2)$, $f(0)$, $f(3)$.

- $f(-2) = (-2)^2 = 4$ (since $-2 < 0$)
- $f(0) = 2(0) + 1 = 1$ (since $0 \geq 0$)
- $f(3) = 2(3) + 1 = 7$ (since $3 \geq 0$)

Solution : Example 2

$$g(x) = \begin{cases} |x| & \text{if } x \leq 1 \\ \sqrt{x-1} & \text{if } x > 1 \end{cases}$$

Find $g(-1)$, $g(1)$, $g(5)$.

- $g(-1) = |-1| = 1$ (since $-1 \leq 1$)

- $g(1) = |1| = 1$ (since $1 \leq 1$)
- $g(5) = \sqrt{5-1} = \sqrt{4} = 2$ (since $5 > 1$)

Solution : Example 3

$$h(x) = \begin{cases} \sin x & \text{if } x < 0 \\ \cos x & \text{if } x \geq 0 \end{cases}$$

Find $h(-\pi/2)$, $h(0)$, $h(\pi)$.

- $h(-\pi/2) = \sin(-\pi/2) = -1$ (since $-\pi/2 < 0$)
- $h(0) = \cos(0) = 1$ (since $0 \geq 0$)
- $h(\pi) = \cos(\pi) = -1$ (since $\pi \geq 0$)

Exercise 2.4 (Function Composition)

1. Given $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$, find $(f \circ g)(x)$ and $(g \circ f)(x)$, and determine their domains.
2. $f(x) = \frac{1}{x}$, $g(x) = x - 2$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.
3. $f(x) = \ln x$, $g(x) = e^x$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution : Example 1 $f(x) = \sqrt{x}$, $g(x) = x^2 + 1$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

1. $(f \circ g)(x) = f(g(x)) = \sqrt{x^2 + 1}$
   Domain: $x^2 + 1 \geq 0$ (always true) $\Rightarrow \mathbb{R}$
2. $(g \circ f)(x) = g(f(x)) = (\sqrt{x})^2 + 1 = x + 1$
   Domain: $x \geq 0$ (from $\sqrt{x}$) $\Rightarrow [0, +\infty)$

Solution : Example 2 $f(x) = \frac{1}{x}$, $g(x) = x - 2$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

1. $(f \circ g)(x) = f(g(x)) = \frac{1}{x-2}$
   Domain: $x - 2 \neq 0 \Rightarrow x \neq 2 \Rightarrow \mathbb{R} \setminus \{2\}$
2. $(g \circ f)(x) = g(f(x)) = \frac{1}{x} - 2$
   Domain: $x \neq 0 \Rightarrow \mathbb{R} \setminus \{0\}$

Solution : Example 3 $f(x) = \ln x$, $g(x) = e^x$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

1. $(f \circ g)(x) = f(g(x)) = \ln(e^x) = x$
   Domain: $e^x > 0$ (always true) $\Rightarrow \mathbb{R}$

2. $(g \circ f)(x) = g(f(x)) = e^{\ln x} = x$
   Domain: $x > 0$ (from $\ln x$) $\Rightarrow (0, +\infty)$

Exercise 2.5 (Limit Calculation)

Evaluate the limit:

1. $\lim_{x\to 2} \frac{x^2-4}{x-2}$

2. $\lim_{x\to 0} \frac{\sin(3x)}{x}$

3. $\lim_{x\to \infty} \frac{3x^2-2x+1}{2x^2+5x-3}$

Solution : Example 1

$$\lim_{x\to 2} \frac{x^2-4}{x-2}$$

$$\begin{aligned}\lim_{x\to 2} \frac{x^2-4}{x-2} &= \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} \\ &= \lim_{x\to 2}(x+2) = 2+2 = 4\end{aligned}$$

Solution : Example 2

$$\lim_{x\to 0} \frac{\sin(3x)}{x}$$

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(3x)}{x} &= \lim_{x\to 0} \frac{3\sin(3x)}{3x} \\ &= 3 \cdot \lim_{u\to 0} \frac{\sin u}{u} \quad \text{(where } u = 3x\text{)} \\ &= 3 \cdot 1 = 3\end{aligned}$$

Solution : Example 3

$$\lim_{x\to \infty} \frac{3x^2-2x+1}{2x^2+5x-3}$$

$$\begin{aligned}\lim_{x\to \infty} \frac{3x^2-2x+1}{2x^2+5x-3} &= \lim_{x\to \infty} \frac{x^2(3-\frac{2}{x}+\frac{1}{x^2})}{x^2(2+\frac{5}{x}-\frac{3}{x^2})} \\ &= \lim_{x\to \infty} \frac{3-\frac{2}{x}+\frac{1}{x^2}}{2+\frac{5}{x}-\frac{3}{x^2}} = \frac{3}{2}\end{aligned}$$

Exercise 2.6 (Continuity Analysis)

1. Determine if the function is continuous at $x = 1$:

$$f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$$

2. Is g continuous at $x = 2$?

$$g(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 4 & \text{if } x = 2 \\ 2x & \text{if } x > 2 \end{cases}$$

3. Is h continuous at $x = 3$?

$$h(x) = \frac{1}{x-3}$$

Solution : Example 1

$$f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$$

Is f continuous at $x = 1$?

1. Check if limit exists:

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1}(x+1) = 2$$

2. Compare with function value: $f(1) = 3$

3. Since $\lim_{x \to 1} f(x) = 2 \neq f(1) = 3$, the function is **not continuous** at $x = 1$.

Solution : Example 2

$$g(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 4 & \text{if } x = 2 \\ 2x & \text{if } x > 2 \end{cases}$$

Is g continuous at $x = 2$?

1. Left-hand limit: $\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} x^2 = 4$

2. Right-hand limit: $\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} 2x = 4$

3. Function value: $g(2) = 4$

4. Since both limits equal the function value, g is **continuous** at $x = 2$.

Solution : Example 3

$$h(x) = \frac{1}{x-3}$$

Is h continuous at $x = 3$?

1. The function is not defined at $x = 3$ (division by zero)

2. Since $h(3)$ does not exist, h is **not continuous** at $x = 3$

3. There is a vertical asymptote at $x = 3$

Exercise 2.7

Let f be the function defined by:

$$f(x) = \frac{\sqrt{1+x} - \sqrt{1+x^2}}{x}$$

1) Find the definition set $\mathcal{D}_f$ of the function f.

2) Calculate $\lim_{x \to 0} f(x)$, is it extendable continuously over $\mathbb{R}$?

Solution: Exercise 2.7

$$f(x) = \frac{\sqrt{1+x} - \sqrt{1+x^2}}{x}$$

1) Domain $\mathcal{D}_f$:

- $\sqrt{1+x}$ requires $1 + x \geq 0 \Rightarrow x \geq -1$
- $\sqrt{1+x^2}$ is defined for all $x \in \mathbb{R}$
- Denominator $x \neq 0$
- Therefore: $\mathcal{D}_f = [-1, 0) \cup (0, +\infty)$

2) Limit at 0:

$$\begin{aligned}
\lim_{x\to 0} f(x) &= \lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1+x^2}}{x} \\
&= \lim_{x\to 0} \frac{(1+x)-(1+x^2)}{x(\sqrt{1+x}+\sqrt{1+x^2})} \\
&= \lim_{x\to 0} \frac{x-x^2}{x(\sqrt{1+x}+\sqrt{1+x^2})} \\
&= \lim_{x\to 0} \frac{1-x}{\sqrt{1+x}+\sqrt{1+x^2}} = \frac{1}{2}
\end{aligned}$$

Since the limit exists and is finite, f can be extended continuously at 0 by defining $f(0) = \frac{1}{2}$.

Exercise 2.8

Let the function g defined on $\mathbb{R}$ be as follows:

$$g(x) = \begin{cases} \frac{1}{\ln|x|} & \text{if } x \notin \{0,-1,1\} \\ 0 & \text{if } x = 0,-1,1 \end{cases}$$

At which points is the function g continuous?

Solution: Exercise 2.8

$$g(x) = \begin{cases} \frac{1}{\ln|x|} & \text{if } x \notin \{0,-1,1\} \\ 0 & \text{if } x = 0,-1,1 \end{cases}$$

1. At $x = 0$: $\lim_{x\to 0} \frac{1}{\ln|x|} = 0$, so continuous at 0
2. At $x = 1$: $\lim_{x\to 1^+} \frac{1}{\ln x} = +\infty$, $\lim_{x\to 1^-} \frac{1}{\ln|x|} = -\infty$, so discontinuous at 1
3. At $x = -1$: $\lim_{x\to -1^+} \frac{1}{\ln|x|} = -\infty$, $\lim_{x\to -1^-} \frac{1}{\ln|x|} = +\infty$, so discontinuous at -1
4. For $x \notin \{-1,0,1\}$: g is continuous as composition of continuous functions
5. Conclusion: g is continuous on $\mathbb{R} \setminus \{-1,1\}$

Exercise 2.9

1) Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined as follows

$$f(x) = \begin{cases} (ax)^2 & \text{if } x \leq 1, \\ a\sin(\frac{\pi}{2}x) & \text{if } x > 1 \end{cases}$$

where $a \in \mathbb{R}$ is a real constant. What are the values of a for the function f to be continuous?

2) Find all values of the constant $\alpha, \beta, \gamma \in \mathbb{R}$ such that the following function $g : \mathbb{R} \to \mathbb{R}$ is continuous:

$$g(x) = \begin{cases} 1 & \text{if } x \leq 0, \\ \alpha e^{-x} + \beta e^{x} + \gamma x(e^{x} - e^{-x}) & \text{if } 0 < x < 1, \\ e^{2-x} & \text{if } x \geq 1. \end{cases}$$

Solution: Exercise 2.9

1) For f to be continuous at $x = 1$:

$$\lim_{x \to 1^-} f(x) = (a \cdot 1)^2 = a^2$$
$$\lim_{x \to 1^+} f(x) = a \sin\left(\frac{\pi}{2} \cdot 1\right) = a \sin\left(\frac{\pi}{2}\right) = a$$
$$\text{Continuity requires: } a^2 = a \Rightarrow a(a-1) = 0 \Rightarrow a = 0 \text{ or } a = 1$$

2) For g to be continuous:

- At $x = 0$: $\lim_{x \to 0^+} g(x) = \alpha + \beta = g(0) = 1$
- At $x = 1$: $\lim_{x \to 1^-} g(x) = \alpha e^{-1} + \beta e + \gamma(e - e^{-1}) = g(1) = e$

So we have the system:

$$\alpha + \beta = 1$$
$$\alpha e^{-1} + \beta e + \gamma(e - e^{-1}) = e$$

Expressing $\beta = 1 - \alpha$ and substituting:

$$\alpha e^{-1} + (1 - \alpha)e + \gamma(e - e^{-1}) = e$$
$$\alpha(e^{-1} - e) + e + \gamma(e - e^{-1}) = e$$
$$\alpha(e^{-1} - e) + \gamma(e - e^{-1}) = 0$$
$$(\gamma - \alpha)(e - e^{-1}) = 0 \Rightarrow \gamma = \alpha$$

So the solutions are: $\alpha \in \mathbb{R}$, $\beta = 1 - \alpha$, $\gamma = \alpha$

Exercise 2.10

Let the function f defined on $\mathbb{R}\backslash\{-1\}$ as follows:

$$f(x) = \frac{1+x}{x^3+1}.$$

1) Prove that we can extend the function f by continuing at the point -1.

2) Find the value taken at -1 for this extension.

Solution: Exercise 2.10

$$f(x) = \frac{1+x}{x^3+1}$$

1) Factor denominator: $x^3 + 1 = (x+1)(x^2 - x + 1)$

$$f(x) = \frac{1+x}{(x+1)(x^2-x+1)} = \frac{1}{x^2-x+1} \quad \text{for } x \neq -1$$

Since $\frac{1}{x^2-x+1}$ is defined and continuous at $x = -1$, f can be extended continuously.

2) The continuous extension at -1 takes value:

$$f(-1) = \frac{1}{(-1)^2 - (-1) + 1} = \frac{1}{1+1+1} = \frac{1}{3}$$

Exercise 2.11

Are the following functions differentiable at 0?

$$f(x) = \frac{x}{1+|x|}, \quad g(x) = \begin{cases} x\sin(x)\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if} x = 0. \end{cases}, \qquad h(x) = |x|\sin x.$$

Solution: Exercise 2.11

1. For $f(x) = \frac{x}{1+|x|}$:

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{\frac{h}{1+|h|}}{h}$$
$$= \lim_{h\to 0} \frac{1}{1+|h|} = 1$$

So f is differentiable at 0 with $f'(0) = 1$.

2. For $g(x) = x\sin(x)\sin(1/x)$:

$$g'(0) = \lim_{h\to 0} \frac{h\sin(h)\sin(1/h)}{h} = \lim_{h\to 0} \sin(h)\sin(1/h) = 0$$

So g is differentiable at 0 with $g'(0) = 0$.

3. For $h(x) = |x|\sin x$:

$$h'_+(0) = \lim_{h\to 0^+} \frac{|h|\sin h - 0}{h} = \lim_{h\to 0^+} \frac{h\sin h}{h} = 0$$
$$h'_-(0) = \lim_{h\to 0^-} \frac{|h|\sin h - 0}{h} = \lim_{h\to 0^-} \frac{-h\sin h}{h} = 0$$

So h is differentiable at 0 with $h'(0) = 0$.

Exercise 2.12

Find $a, b \in \mathbb{R}$ such that the function f defined on $\mathbb{R}_+$ is as follows:

$$f(x) = \begin{cases} \sqrt{x} & \text{if} \quad 0 \le x \le 1, \\ ax^2 + bx + 1 & \text{if} \quad x > 1, \end{cases}$$

differentiable at 1.

Solution: Exercise 2.12

$$f(x) = \begin{cases} \sqrt{x} & 0 \le x \le 1 \\ ax^2 + bx + 1 & x > 1 \end{cases}$$

1. Continuity at $x = 1$: $\sqrt{1} = a(1)^2 + b(1) + 1 \Rightarrow 1 = a + b + 1 \Rightarrow a + b = 0$

2. Differentiability at $x = 1$:

$$f'_-(1) = \frac{1}{2\sqrt{x}}\Big|_{x=1} = \frac{1}{2}$$
$$f'_+(1) = 2ax + b\Big|_{x=1} = 2a + b$$

Equal derivatives: $2a + b = \frac{1}{2}$

3. Solve system:

$$a + b = 0$$
$$2a + b = \frac{1}{2}$$

Subtracting: $a = \frac{1}{2}$, then $b = -\frac{1}{2}$

Exercise 2.13

Study the differentiability of the following functions on $\mathbb{R}$:

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad g(x) = \begin{cases} x^3 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Solution: Exercise 2.13

1. For $f(x) = x^2 \sin(\frac{1}{x})$:

$$f'(0) = \lim_{h \to 0} \frac{h^2 \sin(1/h)}{h} = \lim_{h \to 0} h \sin(1/h) = 0$$

So f is differentiable at 0.

2. For $g(x) = x^3 \sin(\frac{1}{x})$:

$$g'(0) = \lim_{h \to 0} \frac{h^3 \sin(1/h)}{h} = \lim_{h \to 0} h^2 \sin(1/h) = 0$$

So g is differentiable at 0.

Both functions are differentiable on $\mathbb{R}$.

Exercise 2.14

In each case, find the definition set of the function and then its derivative:

1) $f(x) = 4x^3 - 5x^2 + x - 1,$

2) $f(x) = 5x^3 - \dfrac{1}{x} + 3\sqrt{x},$

3) $f(x) = (x^2 + 1)(x^3 - 2x),$

4) $f(x) = \dfrac{2x^2 - 3}{x^2 + 7},$

5) $f(x) = \dfrac{2x - 1}{x + 1},$

6) $f(x) = -x + 2 + \dfrac{2}{3x},$

7) $f(x) = \dfrac{1}{x + x^2},$

8) $f(x) = (2x + 1)^2,$

9) $f(x) = \sqrt{x}(5x - 3).$

Solution: Exercise 2.14

1) $f(x) = 4x^3 - 5x^2 + x - 1$, $\mathcal{D}_f = \mathbb{R}$
$f'(x) = 12x^2 - 10x + 1$

2) $f(x) = 5x^3 - \frac{1}{x} + 3\sqrt{x}$, $\mathcal{D}_f = (0, +\infty)$
$f'(x) = 15x^2 + \frac{1}{x^2} + \frac{3}{2\sqrt{x}}$

3) $f(x) = (x^2 + 1)(x^3 - 2x)$, $\mathcal{D}_f = \mathbb{R}$
$f'(x) = 2x(x^3 - 2x) + (x^2 + 1)(3x^2 - 2) = 5x^4 - 3x^2 - 2$

4) $f(x) = \frac{2x^2-3}{x^2+7}$, $\mathcal{D}_f = \mathbb{R}$
$f'(x) = \frac{4x(x^2+7)-(2x^2-3)(2x)}{(x^2+7)^2} = \frac{34x}{(x^2+7)^2}$

5) $f(x) = \frac{2x-1}{x+1}$, $\mathcal{D}_f = \mathbb{R} \setminus \{-1\}$
$f'(x) = \frac{2(x+1)-(2x-1)}{(x+1)^2} = \frac{3}{(x+1)^2}$

6) $f(x) = -x + 2 + \frac{2}{3x}$, $\mathcal{D}_f = \mathbb{R} \setminus \{0\}$
$f'(x) = -1 - \frac{2}{3x^2}$

7) $f(x) = \frac{1}{x+x^2}$, $\mathcal{D}_f = \mathbb{R} \setminus \{0, -1\}$
$f'(x) = -\frac{1+2x}{(x+x^2)^2}$

8) $f(x) = (2x + 1)^2$, $\mathcal{D}_f = \mathbb{R}$
$f'(x) = 2(2x + 1) \cdot 2 = 8x + 4$

9) $f(x) = \sqrt{x}(5x - 3)$, $\mathcal{D}_f = [0, +\infty)$
$f'(x) = \frac{1}{2\sqrt{x}}(5x - 3) + \sqrt{x} \cdot 5 = \frac{15x-3}{2\sqrt{x}}$

Exercise 2.15 (Derivative using Definition)

Use the definition of derivative to find $f'(x), g'(x), h'(x)$:

1. $f(x) = 3x^2 - 2x$.

2. $g(x) = \frac{1}{x}$.

3. $h(x) = \sqrt{x}$.

Solution : Example 1

$$f(x) = 3x^2 - 2x$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{[3(x+h)^2-2(x+h)]-[3x^2-2x]}{h} \\ &= \lim_{h\to 0} \frac{3(x^2+2xh+h^2)-2x-2h-3x^2+2x}{h} \\ &= \lim_{h\to 0} \frac{6xh+3h^2-2h}{h} \\ &= \lim_{h\to 0}(6x+3h-2) = 6x-2 \end{aligned}$$

Solution : Example 2

$$g(x) = \frac{1}{x}$$

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} \\ &= \lim_{h\to 0} \frac{x-(x+h)}{hx(x+h)} \\ &= \lim_{h\to 0} \frac{-h}{hx(x+h)} = \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2} \end{aligned}$$

Solution : Example 3

$$h(x) = \sqrt{x}$$

$$\begin{aligned} h'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

Exercise 2.16 (Product Rule Application)

Find the derivative of the function using the product rule:

1. $f(x) = (x^2 + 1)(2x^3 - x)$.
2. $g(x) = x^3 \cdot \sin x$.
3. $h(x) = e^x \cdot \ln x$.

Solution : Example 1

$$f(x) = (x^2 + 1)(2x^3 - x)$$

Let $u(x) = x^2 + 1$, $v(x) = 2x^3 - x$
$u'(x) = 2x$, $v'(x) = 6x^2 - 1$
Using product rule: $(uv)' = u'v + uv'$

$$\begin{aligned} f'(x) &= (2x)(2x^3 - x) + (x^2 + 1)(6x^2 - 1) \\ &= 4x^4 - 2x^2 + 6x^4 - x^2 + 6x^2 - 1 \\ &= 10x^4 + 3x^2 - 1 \end{aligned}$$

Solution : Example 2

$$g(x) = x^3 \cdot \sin x$$

Let $u(x) = x^3$, $v(x) = \sin x$
$u'(x) = 3x^2$, $v'(x) = \cos x$

$$\begin{aligned} g'(x) &= (3x^2)(\sin x) + (x^3)(\cos x) \\ &= 3x^2 \sin x + x^3 \cos x \end{aligned}$$

Solution : Example 3

$$h(x) = e^x \cdot \ln x$$

Let $u(x) = e^x$, $v(x) = \ln x$
$u'(x) = e^x$, $v'(x) = \frac{1}{x}$

$$\begin{aligned} h'(x) &= (e^x)(\ln x) + (e^x)\left(\frac{1}{x}\right) \\ &= e^x \left(\ln x + \frac{1}{x}\right) \end{aligned}$$

Exercise 2.17 (Chain Rule Application)

Find the derivative using chain rule of:

1. $f(x) = \sqrt{3x^2 + 2x}$.
2. $g(x) = \sin(2x^3 - x)$.
3. $h(x) = e^{x^2+3x}$.

Solution : Example 1

$$f(x) = \sqrt{3x^2 + 2x}$$

Let $u = 3x^2 + 2x$, then $f(x) = \sqrt{u} = u^{1/2}$
$f'(x) = \frac{1}{2}u^{-1/2} \cdot u' = \frac{1}{2\sqrt{3x^2+2x}} \cdot (6x + 2)$
$f'(x) = \frac{3x+1}{\sqrt{3x^2+2x}}$

Solution : Example 2

$$g(x) = \sin(2x^3 - x)$$

Let $u = 2x^3 - x$, then $g(x) = \sin u$
$g'(x) = \cos u \cdot u' = \cos(2x^3 - x) \cdot (6x^2 - 1)$

Solution : Example 3

$$h(x) = e^{x^2+3x}$$

Let $u = x^2 + 3x$, then $h(x) = e^u$
$h'(x) = e^u \cdot u' = e^{x^2+3x} \cdot (2x + 3)$

Exercise 2.18 (Higher Order Derivatives)

Find the first three derivatives of

1. $f(x) = \sin(2x) + e^{3x}$.
2. $g(x) = x^4 - 3x^3 + 2x^2 - x + 5$.
3. $h(x) = \ln(2x + 1)$.

Solution : Example 1

$$f(x) = \sin(2x) + e^{3x}$$

$$f'(x) = 2\cos(2x) + 3e^{3x}$$
$$f''(x) = -4\sin(2x) + 9e^{3x}$$
$$f'''(x) = -8\cos(2x) + 27e^{3x}$$

Solution : Example 2

$$g(x) = x^4 - 3x^3 + 2x^2 - x + 5$$

$$g'(x) = 4x^3 - 9x^2 + 4x - 1$$
$$g''(x) = 12x^2 - 18x + 4$$
$$g'''(x) = 24x - 18$$

Solution : Example 3

$$h(x) = \ln(2x + 1)$$

$$h'(x) = \frac{2}{2x+1} = 2(2x+1)^{-1}$$
$$h''(x) = 2 \cdot (-1)(2x+1)^{-2} \cdot 2 = -\frac{4}{(2x+1)^2}$$
$$h'''(x) = -4 \cdot (-2)(2x+1)^{-3} \cdot 2 = \frac{16}{(2x+1)^3}$$

Exercise 2.19 (Function Monotony)

Determine intervals of increase and decrease:

1. $f(x) = x^3 - 3x^2 - 9x + 5$.
2. $g(x) = x^4 - 8x^2 + 16$.
3. $h(x) = \frac{x}{x^2+1}$.

Solution : Example 1

$$f(x) = x^3 - 3x^2 - 9x + 5$$

1. Find derivative: $f'(x) = 3x^2 - 6x - 9$

2. Factor: $f'(x) = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$

3. Critical points: $x = -1$, $x = 3$

4. Sign analysis:

   - $f'(x) > 0$ when $x < -1$ or $x > 3 \Rightarrow$ increasing on $(-\infty, -1) \cup (3, \infty)$
   - $f'(x) < 0$ when $-1 < x < 3 \Rightarrow$ decreasing on $(-1, 3)$

Solution : Example 2

$$g(x) = x^4 - 8x^2 + 16$$

Solution:

1. Find derivative: $g'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x-2)(x+2)$

2. Critical points: $x = -2$, $x = 0$, $x = 2$

3. Sign analysis:

   - $g'(x) > 0$ when $-2 < x < 0$ or $x > 2 \Rightarrow$ increasing on $(-2, 0) \cup (2, \infty)$
   - $g'(x) < 0$ when $x < -2$ or $0 < x < 2 \Rightarrow$ decreasing on $(-\infty, -2) \cup (0, 2)$

Solution : Example 3

$$h(x) = \frac{x}{x^2 + 1}$$

1. Find derivative using quotient rule: $h'(x) = \frac{(1)(x^2+1)-(x)(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$

2. Critical points: $1 - x^2 = 0 \Rightarrow x = \pm 1$

3. Sign analysis:

   - $h'(x) > 0$ when $1 - x^2 > 0 \Rightarrow -1 < x < 1 \Rightarrow$ increasing on $(-1, 1)$
   - $h'(x) < 0$ when $x < -1$ or $x > 1 \Rightarrow$ decreasing on $(-\infty, -1) \cup (1, \infty)$

Exercise 2.20 (Bounded Function Analysis)

Analyze boundedness and find extrema:

1. Show that $f(x) = \frac{1}{x^2+1}$ is bounded and find its maximum and minimum values.

2. The same thing about $g(x) = \sin x + 2$.

3. The same thing about $h(x) = \frac{x^2}{x^2+4}$.

Solution : Example 1

$$f(x) = \frac{1}{x^2+1}$$

1. Since $x^2 + 1 \geq 1$ for all $x \in \mathbb{R}$, we have:
$$0 < \frac{1}{x^2+1} \leq 1$$
2. Maximum value: $f(0) = 1$ (achieved at $x = 0$)
3. Minimum value: approaches 0 as $x \to \pm\infty$ (but never reaches 0)
4. Therefore, f is bounded with $0 < f(x) \leq 1$

Solution : Example 2

$$g(x) = \sin x + 2$$

1. Since $-1 \leq \sin x \leq 1$, we have:
$$1 \leq \sin x + 2 \leq 3$$
2. Maximum value: 3 (achieved when $\sin x = 1$)
3. Minimum value: 1 (achieved when $\sin x = -1$)
4. Therefore, g is bounded with $1 \leq g(x) \leq 3$

Solution : Example 3

$$h(x) = \frac{x^2}{x^2+4}$$

1. Since $x^2 + 4 > x^2 \geq 0$ for all $x \in \mathbb{R}$, we have:
$$0 \leq \frac{x^2}{x^2+4} < 1$$
2. Maximum value: approaches 1 as $x \to \pm\infty$ (but never reaches 1)
3. Minimum value: $h(0) = 0$ (achieved at $x = 0$)
4. Therefore, h is bounded with $0 \leq h(x) < 1$