Chapter 4

Integral Calculus

4.1 Integral Calculus

Integral calculus is the study of integrals and their properties, forming one of the two main branches of calculus (along with differential calculus). There are two fundamental types of integrals: indefinite and definite integrals.

Let $I = [a, b]$ is a non-empty open interval in $\mathbb{R}$ and let the function $f : I \to \mathbb{R}$. We call F a primitive function of f on I such that:

$$F : I \to \mathbb{R}$$

satisfying:

-1 F can be derived in the open interval I.

-2

$$\forall x \in I, \quad F'(x) = f(x)$$

4.1.1 Indefinite Integrals (Antiderivatives)

Definition 4.1 (Antiderivative)

A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all $x \in I$. The **indefinite integral** is denoted by:

$$\int f(x)\,dx = F(x) + C$$

where C is the constant of integration.

Example 4.1 (Basic Integrals)

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$
$$\int \frac{1}{x}\,dx = \ln|x| + C$$
$$\int e^x\,dx = e^x + C$$
$$\int \sin x\,dx = -\cos x + C$$
$$\int \cos x\,dx = \sin x + C$$

4.1.2 Definite Integrals

Definition 4.2 (Riemann Integral)

The **definite integral** of f from a to b is defined as:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i^* \in [x_{i-1}, x_i]$.

Theorem 4.1 (Fundamental Theorem of Calculus)

Let f be continuous on $[a, b]$.

1. If $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$
2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f

Example 4.2 (Definite Integral Calculation)

Compute $\int_0^1 x^2\,dx$:

$$\int x^2\,dx = \frac{x^3}{3} + C$$
$$\int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3} - 0 = \boxed{\frac{1}{3}}$$

Theorem 4.2

If F is a primitive function of $f : I \to \mathbb{R}$ on I, then F is continuous on I.

Theorem 4.3

Let $f : I \to \mathbb{R}$ has a primitive function on I. Then : the set of primitive functions of f is:

$$\{F + c, c \in \mathbb{R}\},$$

where, F is a primitive function of f.

All primitive functions of f are obtained by shifting any primitive function of f by a constant. We denote by $\int f(t)dt$ the primitive function of f and we write:

$$F(x) = \int f(x)dx.$$

4.1.3 Definite integral

There are two types of integrals: definite integrals and indefinite integrals. Let $f : [a, b] \to \mathbb{R}$ the continues function on $[a, b]$ such that $b \geq a$.

Integration can be defined in another way that is more used to find constant values for the integrals through following theorem:

Theorem 4.4

Let $F : [a, b] \to \mathbb{R}$ be the function defined as:

$$F(x) = \int_a^x f(t)dt$$

a primitive function of f means that F derivable and satisfying :

$$F'(x) = f(x), \forall x \in [a, b].$$

Definition 4.3

The definite integral of f is a number which represents the area under the curve $f(x)$ from $x = a$ to $x = b$ denoted by:

$$\int_a^b f(x)\, dx$$

The real number $F(b) - F(a)$ where F the primitive function of f and we write:

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Example 4.3

Let's calculate the following integrals:

-1 For $f(x) = e^x$ let $F(x) = e^x$ be its primitive function, then

$$\int_0^1 e^x\ dx = \left[e^x\right]_0^1 = e^1 - e^0 = e - 1.$$

-2 For $g(x) = x^2$ let $G(x) = \frac{x^3}{3}$ be its primitive function, then

$$\int_0^1 x^2\ dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

-3

$$\int_a^x \cos t\ dt = \left[\sin t\right]_{t=a}^{t=x} = \sin x - \sin a$$

is a primitive function of $\cos x$.

-4 If the function is odd, then its primitive function is be an even function (proved later). We conclude that:

$$\int_{-a}^{a} f(t)\, dt = 0.$$

4.2 Properties of integrals

The three main properties to integral calculus are the relation Chasles, positivity and linearity of integral.

4.2.1 Chasles relation

Proposition 4.2.1. *Let $a < c < b$. If f integrable on $[a, c]$ and $[c, b]$ then f integrable on $[a, b]$, and we have:*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

We have the following proprieties, for $a = b$:

$$\int_a^a f(x)dx = 0.$$

and for $a < b$:

$$\int_b^a f(x)dx = -\int_a^b f(x)dx.$$

Example 4.4

We have:

$$\begin{aligned} \int_1^3 x^2dx &= \left[\frac{x^3}{3}\right]_1^3 = \frac{27}{3} - \frac{1}{3} = \frac{26}{3} \\ \int_3^1 x^2dx &= \left[\frac{x^3}{3}\right]_3^1 = \frac{1}{3} - \frac{27}{3} = -\frac{26}{3} \end{aligned}$$

$$\int_1^3 x^2dx = -\int_3^1 x^2dx.$$

4.2.2 Positivity of integration

Proposition 4.2.2. *Let $a \leq b$ two real numbers, f and g two functions have a primitive functions on $[a,b]$. If $f \leq g$ then:*

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

In particular, the integral of a positive function is positive: If $f \geq 0$ then:

$$\int_a^b f(x)\, dx \geq 0.$$

4.2.3 Linearity of integration

Proposition 4.2.3. *Let f and g two functions have a primitive on $[a,b]$*

-1 *then $f+g$ a function integrable and*

$$\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

-2 *For all real number λ the function λf is integrable and we have:*

$$\int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx.$$

From these first two points we have the linearity of integration: For all real numbers λ and μ we have:

$$\int_a^b \big(\lambda f(x) + \mu g(x)\big)dx = \lambda \int_a^b f(x)dx + \mu \int_a^b g(x)dx.$$

Remark 4.2.1.

(1 *If f and g are integrable functions on $[a,b]$ then most of the time we have:*

$$\int_a^b (fg)(x)\, dx \neq \left(\int_a^b f(x)\, dx\right)\left(\int_a^b g(x)\, dx\right).$$

(2 *If f is an integrable function on $[a,b]$ then $|f|$ is also an integrable function on $[a,b]$ and we have:*

$$\left|\int_a^b f(x)\, dx\right| \leq \int_a^b \big|f(x)\big|\, dx.$$

Example 4.5

We have:

$$\int_0^1 \left(7x^2 - e^x\right)\, dx = 7\int_0^1 x^2\, dx \quad - \quad \int_0^1 e^x\, dx = 7\frac{1}{3} \; - \; (e-1) = \frac{10}{3} - e$$

Using the calculations we saw earlier, we find:

$$\int_0^1 x^2\, dx = \frac{1}{3}$$

and

$$\int_0^1 e^x dx = e - 1.$$

Example 4.6

Let

$$I_n = \int_1^n \frac{\sin(nx)}{1+x^n}\, dx$$

Let's prove that $I_n \to 0$ for $n \to +\infty$.

$$|I_n| = \left|\int_1^n \frac{\sin(nx)}{1+x^n}\, dx\right| \leq \int_1^n \frac{|\sin(nx)|}{1+x^n}\, dx \leq \int_1^n \frac{1}{1+x^n}\, dx \leq \int_1^n \frac{1}{x^n}\, dx$$

It remains only for us to calculate this last integral

$$\int_1^n \frac{1}{x^n}\, dx = \int_1^n x^{-n}\, dx = \left[\frac{x^{-n+1}}{-n+1}\right]_1^n = \frac{n^{-n+1}}{-n+1} - \frac{1}{-n+1} \xrightarrow[n\to+\infty]{} 0$$

because $n^{-n+1} \to 0$ and $\frac{1}{-n+1} \to 0$ for $n \to +\infty$.

Remark 4.2.2. *We note that even if $f \cdot g$ is an integrable function, in general we have:*

$$\int_a^b (fg)(x)dx \neq \big(\int_a^b f(x)dx\big)\big(\int_a^b g(x)dx\big).$$

For example, let the functions f and g be defined as follows:

$$f : [0,1] \to \mathbb{R}, f(x) = \begin{cases} 1 & si \quad x \in [0, \frac{1}{2}[\\ 0 & si\ non. \end{cases}$$

and

$$g : [0,1] \to \mathbb{R}, g(x) = \begin{cases} 1 & si \quad x \in [\frac{1}{2}, 1[\\ 0 & si\ non. \end{cases}$$

Hence $f(x) \cdot g(x) = 0$ *for each* $x \in [0,1]$, *then:*

$$\int_0^1 f(x)g(x)dx = 0$$

although

$$\int_0^1 f(x)\, dx = \frac{1}{2} \quad and \quad \int_0^1 g(x)\, dx = \frac{1}{2}.$$

4.3 Primitive of usual functions

1. $\int e^x dx = e^x + c \quad$ on $\mathbb{R}$
2. $\int \cos x dx = \sin x + c \quad$ on $\mathbb{R}$
3. $\int \sin x dx = -\cos x + c \quad$ on $\mathbb{R}$
4. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c \quad , n \in \mathbb{N} \quad$ on $\mathbb{R}$
5. $\int x^\alpha dx = \dfrac{x^{\alpha+1}}{\alpha+1} + c \quad , \alpha \in \mathbb{R} \setminus \{-1\}$ on $]0, +\infty[$
6. $\int \dfrac{1}{x} dx = \ln |x| + c \quad$ on $]0, +\infty[$ or $] - \infty, 0[$
7. $\int \sinh x\ dx = \cosh x + c$, $\int \cosh x\ dx = \sinh x + c \quad$ on $\mathbb{R}$
8. $\int \dfrac{dx}{1+x^2} = \arctan x + c \quad$ on $\mathbb{R}$
9. $\int \dfrac{dx}{\sqrt{1-x^2}} = \begin{cases} \arcsin x + c \\ \dfrac{\pi}{2} - \arccos x + c \end{cases} \quad$ on $] - 1, 1[$
10. $\int \dfrac{dx}{\sqrt{x^2+1}} = \begin{cases} \text{Argsh}(x) + c \\ \ln\left(x + \sqrt{x^2+1}\right) + c \end{cases} \quad$ on $\mathbb{R}$
11. $\int \dfrac{dx}{\sqrt{x^2-1}} = \begin{cases} \text{Argch}(x) + c \\ \ln\left(x + \sqrt{x^2-1}\right) + c \end{cases} \quad$ on $x \in]1, +\infty[$

4.4 Integration methods

Integration is based on finding the primitive function of the function we want to integrate. On November 13, 1675, Gottfried Wilhelm Leibniz demonstrated the first integral for calculating area. Leibniz established the mathematical calculus independently of Isaac Newton, and his mathematical symbols are still in common use since they were first published. There are several methods of integration, including: integration by parts, integration by substitution, integration by changing the variable, ...

4.4.1 Integration per partes

Theorem 4.5

Let u and v two functions of the class $\mathcal{C}^1$ defined on $[a, b]$, then :

$$\int_a^b u(x)\, v'(x)\, dx = [uv]_a^b - \int_a^b u'(x)\, v(x)\, dx.$$

The formula for the fractional integral for the primitive function is the same but without bounds:

$$\int u(x)v'(x)\, dx = [uv] - \int u'(x)v(x)\, dx.$$

Example 4.7 (Parts Integration)

To calculate the integral

$$\int_0^1 xe^x\, dx$$

We put $u(x) = x$ and $v'(x) = e^x$. We know that the function $u'(x) = 1$ is the derivative of the function $u(x)$ and the function $v(x) = e^x$ is the primitive function of v' by using the integration by parts formula we find:

$$\begin{aligned}\int_0^1 xe^x dx &= \int_0^1 u(x)v'(x)\,dx\\ &= [u(x)v(x)]_0^1 - \int_0^1 u'(x)v(x)\,dx\\ &= [xe^x]_0^1 - \int_0^1 1\cdot e^x\,dx\\ &= (1\cdot e^1 - 0\cdot e^0) - [e^x]_0^1\\ &= e - (e^1 - e^0)\\ &= 1\end{aligned}$$

Example 4.8

To calculate the integral

$$\int_1^e x\ln x\,dx.$$

This time we put $u(x) = \ln x$ and $v'(x) = a$. The function $u' = \dfrac{1}{x}$ is the derivative of $u(x)$ and the function $v = \frac{x^2}{2}$ is the primitive of v'. by using the integration by parts formula we find:

$$\begin{aligned}\int_1^e \ln x\cdot x\,dx = \int_1^e uv' = [uv]_1^e - \int_1^e u'v = \left[\ln x\cdot\frac{x^2}{2}\right]_1^e - \int_1^e \frac{1}{x}\frac{x^2}{2}\,dx\\ = \left(\ln e\frac{e^2}{2} - \ln 1\frac{1^2}{2}\right) - \frac{1}{2}\int_1^e x\,dx = \frac{e^2}{2} - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^e\\ = \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2+1}{4}.\end{aligned}$$

Example 4.9

To calculate the integral

$$\int \arcsin x\,dx$$

to finds a primitive function of the $\arcsin(x)$ function

we make it in the form of a product, we put $u(x) = \arcsin(x)$ and $v'(x) = 1$, where we have $u'(x) = \dfrac{1}{\sqrt{1-x^2}}$ and $v(x) = x$ then we use the integration by parts formula we find:

$$\begin{aligned}\int 1\cdot \arcsin(x)dx &= [x\arcsin(x)] - \int \frac{x}{\sqrt{1-x^2}}\,dx\\ &= [x\arcsin(x)] - \left[-\sqrt{1-x^2}\right]\\ &= x\arcsin(x) + \sqrt{1-x^2} + c.\end{aligned}$$

Example 4.10

To calculate the integral

$$\int x^2e^xdx.$$

we put $u(x) = x^2$ and $v'(x) = e^x$. We know that the function $u'(x) = 2x$ is the derivative of $u(x)$ and $v(x) = e^x$ is the primitive function of $v'(x)$ and by using the integration by parts formula we find:

$$\int x^2e^x\,dx = \left[x^2e^x\right] - 2\int xe^x\,dx$$

Re-integrating by parts for the second time on the second part of the previous equations, we find:

$$\int xe^x\,dx = [xe^x] - \int e^x\,dx = (x-1)e^x + c,$$

Finally we find

$$\int x^2e^x\,dx = (x^2-2x+2)e^x + c.$$

4.4.2 Change of variables

Theorem 4.6

Let f be a function defined on $I = [a,b]$ and let the mapping $\varphi : J \to I$ be in class $\mathcal{C}^1$. for all $a, b \in J$ we have:

$$\int_{\varphi(a)}^{\varphi(b)} f(x)dx = \int_a^b f(\varphi(t)) \cdot \varphi'(t)dt$$

if F is a primitive function of f then $F \circ \varphi$ is the primitive function of $(f \circ \varphi) \cdot \varphi'$. in another way

$$\left(\int f(x)\ dx\right) \circ \varphi = \int f(\varphi(t))\, \varphi'(t)\ dt.$$

that is, the primitive function $f(\varphi(t))\varphi'(t)$ results from the combination of f and φ. the statement $\int f(x)\ dx = \int f(\varphi(t))\, \varphi'(t)\ dt$ is actually a change of the variable, or in a simplified form we put

$$x = \varphi(t)$$

after derivation, we find

$$\frac{dx}{dt} = \varphi'(t)$$

i.e.

$$dx = \varphi'(t)\ dt$$

what it gives us:

$$\int_{\varphi(a)}^{\varphi(b)} f(x)dx = \int_a^b f(\varphi(t))\varphi'(t)dt.$$

Example 4.11

Calculate the integral

$$\int_0^{\frac{\pi}{2}} \sin^2(x)\cos(x)\, dx$$

by placing

$$\sin(x) = t \Longrightarrow \sin(x)' = \cos(x) = dt$$

Hence, the bounds of integration change from x to t as follows

$$\begin{aligned} x &= 0 \Longrightarrow t = \sin(0) = 0 \\ x &= \frac{\pi}{2} \Longrightarrow t = \sin\left(\frac{\pi}{2}\right) = 1 \end{aligned}$$

from it we find

$$\begin{aligned} x &= 0 \Longrightarrow \sin(0) = 0 \\ x &= \frac{\pi}{2} \Longrightarrow \sin\left(\frac{\pi}{2}\right) = 1 \end{aligned}$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin^2(x)\cos(x)\,dx &= \int_0^1 t^2 dt \\ &= \frac{1}{3}t^3\Big]_0^1 \\ &= \frac{1}{3}. \end{aligned}$$

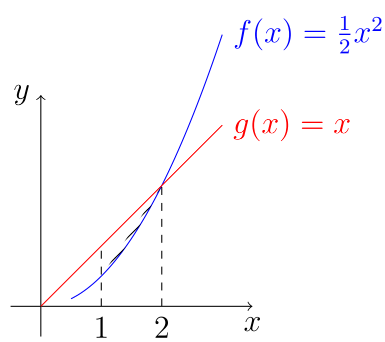
4.4.3 Applications of Integration

Area Between Curves

Theorem 4.7

The area between two curves $f(x)$ and $g(x)$ from a to b is:

$$A = \int_a^b |f(x) - g(x)|\,dx$$



Example 4.12 (Area Calculation)

Find the area between $y = x$ and $y = \frac{1}{2}x^2$ from $x = 1$ to $x = 2$:

$$\begin{aligned} A &= \int_1^2 \left(x - \frac{1}{2}x^2\right) dx \\ &= \left[\frac{1}{2}x^2 - \frac{1}{6}x^3\right]_1^2 \\ &= \left(2 - \frac{8}{6}\right) - \left(\frac{1}{2} - \frac{1}{6}\right) = \boxed{\frac{1}{6}} \end{aligned}$$

Volumes of Revolution

Theorem 4.8 (Disk Method)

The volume generated by rotating $y = f(x)$ about the x-axis from a to b is:

$$V = \pi \int_a^b [f(x)]^2 \, dx$$

Example 4.13 (Volume Calculation)

Find the volume of the solid formed by rotating $y = \sqrt{x}$ from 0 to 4 about the x-axis:

$$\begin{aligned} V &= \pi \int_0^4 (\sqrt{x})^2 \, dx = \pi \int_0^4 x \, dx \\ &= \pi \left[\frac{x^2}{2}\right]_0^4 = \pi\,(8 - 0) = \boxed{8\pi} \end{aligned}$$

4.4.4 Integration Tables

Integral Form	**Antiderivative**
$\int \frac{1}{x^2+a^2} \, dx$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{1}{\sqrt{a^2-x^2}} \, dx$	$\sin^{-1}\left(\frac{x}{a}\right) + C$
$\int \ln x \, dx$	$x \ln x - x + C$
$\int \sin^2 x \, dx$	$\frac{x}{2} - \frac{\sin 2x}{4} + C$
$\int \sec x \, dx$	$\ln\lvert\sec x + tanx\rvert + C$

Differential equations are the best way to describe most engineering, mathematical and scientific issues alike, such as describing heat transfer processes or fluid flow, wave motion and electronic circuits and using them in issues of the structural structures of matter or the mathematical description of chemical reactions.

4.5 Basic Concepts

This chapter includes a set of definitions and concepts in differential equations, the most important of which are:

Definition 4.4

A differential equation is every equation that contains differentials or derivatives of one or more functions with respect to variables and is of the form:

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0. \qquad (E)$$

Example 4.14

$$\frac{dx}{dy}z + ydx = u$$

The differential equation is classified into: 1- Ordinary differential equation: It is a differential equation that contains derivatives or ordinary differentials of one or more variables.

Example 4.15

$$ydx + xdy = e^z.$$

2- Partial differential equation: It is a differential equation that contains derivatives or partial differentials of one or more variables.

Example 4.16

$$\frac{\partial x}{\partial y} = zx.$$

3- For the linear ordinary differential equation: it is the equation that is linear with respect to each of the function(s) and their derivatives does not contain their products.

4- Linear partial differential equation: It is the equation that is linear with respect to the partial derivatives of the existing function or functions.

Remark 4.5.1. 1- *The order of an equation is the order of the highest derivative present in it.*

2- *The differential equation can be converted from one form to another to facilitate its solution.*

4.5.1 Order and Degree

Order

Definition 4.5

The order of a differential equation: is the order of the highest derivative (also known as differential coefficient) present in the equation.

Example 4.17

$$\frac{dy}{dx} + y^3 = \cos(x)$$

Contains only the first derivative $\frac{dy}{dx}$, which is a first order differential equation.

Example 4.18

$$\frac{d^3x}{dx^3} + 3x\frac{dy}{dx} = e^y$$

In this equation, the order of the highest derivative is 3 hence, this is a third order differential equation.

Degree

Definition 4.6

The degree of the differential equation is represented by the power of the highest order derivative in the given differential equation.

The differential equation must be a polynomial equation in derivatives for the degree to be defined.