3 Numerical Series

3.1 Numerical Series

3.1.1 Definitions and Basic Concepts

Definition 3.1.1 (Numerical Series) —
*Let (u_n) be a sequence of real numbers. The **numerical series** $\sum u_n$ is defined through its partial sums:*

$$S_n = u_0 + u_1 + u_2 + \cdots + u_n = \sum_{k=0}^{n} u_k$$

*We say the series **converges** if $\lim_{n\to\infty} S_n = S$ exists and is finite, and write:*

$$\sum_{n=0}^{\infty} u_n = S$$

*Otherwise, the series **diverges**.*

Example 3.1.1 (Telescoping Series) —
Consider $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$. Using partial fractions:

$$\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

The partial sum telescopes:

$$\begin{aligned} S_n &= \frac{1}{2}\left[\left(1-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\right] \\ &= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) \end{aligned}$$

Thus $\lim_{n\to\infty} S_n = \frac{1}{2}$, so the series converges to $\frac{1}{2}$.

Example 3.1.2 (Divergent Series) —
The series $\sum_{n=0}^{\infty}(-1)^n$ has partial sums:

$$S_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Since the limit doesn't exist, the series diverges.

3.1.2 Geometric Series

Definition 3.1.2 (Geometric Series) —
A series of the form $\sum_{n=0}^{\infty} aq^n$ *where* $a, q \in \mathbb{R}$.

- *Partial sum:* $S_n = a\frac{1-q^{n+1}}{1-q}$ *for* $q \neq 1$
- *The series converges if and only if* $|q| < 1$*, with sum* $\frac{a}{1-q}$
- *Diverges if* $|q| \geq 1$

Example 3.1.3 —
$\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges to 2 since $a = 1$, $q = \frac{1}{2}$, and $\frac{1}{1-1/2} = 2$.

3.1.3 Series with Positive Terms

Definition 3.1.3 —
A series $\sum u_n$ *has* ***positive terms*** *if* $u_n \geq 0$ *for all* n.

For positive term series, the sequence of partial sums (S_n) is increasing. Therefore:

Theorem 3.1.1 *A positive term series converges if and only if its partial sums are bounded above.*

3.1.4 Riemann Series and Comparison Tests

Definition 3.1.4 (Riemann Series) —
The series $\sum \frac{1}{n^\alpha}$ *where* $\alpha \in \mathbb{R}$.

Theorem 3.1.2 *The Riemann series* $\sum \frac{1}{n^\alpha}$ *converges if and only if* $\alpha > 1$.

Example 3.1.4
- $\sum \frac{1}{n^2}$ converges ($\alpha = 2 > 1$)
- $\sum \frac{1}{n}$ diverges ($\alpha = 1 \leq 1$)
- $\sum \frac{1}{\sqrt{n}}$ diverges ($\alpha = 0.5 < 1$)

Theorem 3.1.3 (Comparison Test) —
Let $0 \leq u_n \leq v_n$ *for all sufficiently large* n.

- *If* $\sum v_n$ *converges, then* $\sum u_n$ *converges*
- *If* $\sum u_n$ *diverges, then* $\sum v_n$ *diverges*

Example 3.1.5 —

- $\sum \frac{|\cos(n)^n|}{n^2} \leq \sum \frac{1}{n^2}$ converges
- $\sum \sin\left(\frac{\pi}{2^n}\right) \leq \sum \frac{\pi}{2^n}$ converges

3.1.5 Equivalent Sequences and Limit Comparison

Definition 3.1.5 (Equivalent Sequences) —
Two sequences (u_n) and (v_n) (with $v_n \neq 0$) are ***equivalent*** *if:*

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 1$$

We write $u_n \sim v_n$.

Theorem 3.1.4 —
If $u_n \sim v_n$ and both sequences are positive, then $\sum u_n$ and $\sum v_n$ have the same nature (both converge or both diverge).

Example 3.1.6 —
Since $\frac{1}{n(n+1)} \sim \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges, then $\sum \frac{1}{n(n+1)}$ also converges.

3.1.6 Cauchy Criterion

Theorem 3.1.5 (Cauchy Criterion for Series) —
The series $\sum u_n$ converges if and only if:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall m > n \geq N, \left|\sum_{k=n+1}^{m} u_k\right| < \varepsilon$$

Example 3.1.7 (Harmonic Series) —
The harmonic series $\sum \frac{1}{n}$ diverges even though $\lim_{n\to\infty} \frac{1}{n} = 0$. For $m = 2n$:

$$S_{2n} - S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > n \cdot \frac{1}{2n} = \frac{1}{2}$$

So the Cauchy criterion is violated.

3.1.7 Ratio Test (d'Alembert's Rule)

Theorem 3.1.6 (Ratio Test) —
Let $\sum u_n$ be a series with positive terms and suppose $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = L$.

- *If $L < 1$, the series converges*
- *If $L > 1$, the series diverges*
- *If $L = 1$, the test is inconclusive*

Example 3.1.8 —

- $\sum \frac{x^n}{n!}$: $\frac{u_{n+1}}{u_n} = \frac{x}{n+1} \longrightarrow 0$, so converges for all x
- $\sum \frac{2^n}{n!}$: $\frac{u_{n+1}}{u_n} = \frac{2}{n+1} \longrightarrow 0$, so converges
- $\sum \frac{n^2}{(2n)!}$: $\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{(2n+2)(2n+1)n^2} \longrightarrow 0$, so converges

3.1.8 Alternating Series Test

Definition 3.1.6 (Alternating Series) —
A series of the form $\sum(-1)^n a_n$ or $\sum(-1)^{n+1} a_n$ where $a_n > 0$.

Theorem 3.1.7 (Leibniz Test for Alternating Series) —
If (a_n) is a decreasing sequence with $\lim_{n\to\infty} a_n = 0$, then the alternating series $\sum(-1)^n a_n$ converges.

Example 3.1.9 —

- $\sum \frac{(-1)^n}{n}$ converges (alternating harmonic series)
- $\sum \frac{(-1)^n n^2}{1+n}$ diverges since $\lim_{n\to\infty} \frac{n^2}{1+n} \neq 0$

3.1.9 Absolute and Conditional Convergence

Definition 3.1.7 —
A series $\sum u_n$ is:

- ***Absolutely convergent** if $\sum |u_n|$ converges*
- ***Conditionally convergent** if $\sum u_n$ converges but $\sum |u_n|$ diverges*

Theorem 3.1.8 *Absolute convergence implies convergence.*

Example 3.1.10 —

- $\sum \frac{\cos(n\pi)}{n^2} = \sum \frac{(-1)^n}{n^2}$ converges absolutely
- $\sum \frac{\cos(n\pi)}{n} = \sum \frac{(-1)^n}{n}$ converges conditionally

3.2 Function Series

3.2.1 Pointwise and Uniform Convergence

Definition 3.2.1 (Pointwise Convergence) —
*A sequence of functions (f_n) converges **pointwise** to f on A if:*

$$\forall x \in A, \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, |f_n(x) - f(x)| < \varepsilon$$

Definition 3.2.2 (Uniform Convergence) —
*A sequence of functions (f_n) converges **uniformly** to f on A if:*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall x \in A, \forall n \geq N, |f_n(x) - f(x)| < \varepsilon$$

Remark 3.2.1 —
Uniform convergence means the convergence rate is the same for all $x \in A$, while pointwise convergence allows different rates at different points.

Example 3.2.1 —
Let $f_n(x) = x^n$ on $[0, 1]$.

- Pointwise limit: $f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$
- Convergence is not uniform since all f_n are continuous but f is discontinuous

3.2.2 Properties Preserved by Uniform Convergence

Theorem 3.2.1 (Continuity) —
If f_n are continuous on A and $f_n \to f$ uniformly on A, then f is continuous on A.

Theorem 3.2.2 (Integration) —

If f_n are integrable on $[a,b]$ and $f_n \to f$ uniformly on $[a,b]$, then:

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

Theorem 3.2.3 (Differentiation) —
If f_n are differentiable on I, $f_n \to f$ pointwise on I, and $f_n' \to g$ uniformly on I, then f is differentiable and $f' = g$.

Example 3.2.2 (Failure of Term-by-Term Differentiation) —
Let $f_n(x) = \frac{\sin(nx)}{n}$ on $\mathbb{R}$.

- $f_n \to 0$ uniformly since $|f_n(x)| \le \frac{1}{n}$
- $f_n'(x) = \cos(nx)$ does not converge
- So we cannot differentiate term-by-term

3.2.3 Series of Functions

Definition 3.2.3 (Function Series) —
*A **series of functions** is an expression $\sum_{n=1}^{\infty} f_n(x)$.*

Definition 3.2.4 (Normal Convergence) —
*A function series $\sum f_n(x)$ converges **normally** on A if there exists a convergent numerical series $\sum M_n$ such that:*

$$\forall x \in A, |f_n(x)| \le M_n$$

Theorem 3.2.4 —
Normal convergence implies uniform convergence.

Example 3.2.3 —
$\sum \frac{\sin(nx)}{n^2}$ converges normally on $\mathbb{R}$ since:

$$\left|\frac{\sin(nx)}{n^2}\right| \le \frac{1}{n^2} \quad \text{and} \quad \sum \frac{1}{n^2} \text{ converges}$$

3.2.4 Power Series

Definition 3.2.5 (Power Series) —
A series of the form $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ where $a_n, z_0 \in \mathbb{C}$.

Theorem 3.2.5 (Radius of Convergence) —
For every power series $\sum a_n z^n$, there exists $R \in [0, \infty]$ (the radius of convergence) such that:

- *The series converges absolutely for $|z| < R$*
- *The series diverges for $|z| > R$*

Theorem 3.2.6 (Operations on Power Series) —
Within their common interval of convergence:

- *Sum:*
$$\sum (a_n + b_n) z^n = \sum a_n z^n + \sum b_n z^n$$
- *Product:*
$$\left(\sum a_n z^n\right)\left(\sum b_n z^n\right) = \sum c_n z^n$$
where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$
- *Differentiation:*
$$\frac{d}{dz}\left(\sum a_n z^n\right) = \sum n a_n z^{n-1}$$
- *Integration:*
$$\int_0^z \left(\sum a_n t^n\right) dt = \sum \frac{a_n}{n+1} z^{n+1}$$

Example 3.2.4 (Common Power Series) —

- Exponential:
$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$
$R = \infty$
- Sine:
$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$
$R = \infty$
- Cosine:
$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!},$$
$R = \infty$
- Geometric:
$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n,$$
$R = 1$

- Logarithm:

$$\ln(1+z) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n},$$

$R = 1$

3.2.5 Fourier Series

Definition 3.2.6 (Fourier Series) —
A series of the form:

$$a_0 + \sum_{n=1}^{\infty}[a_n\cos(n\omega t) + b_n\sin(n\omega t)]$$

where $\omega = \frac{2\pi}{T}$ *for some period* T.

Theorem 3.2.7 (Fourier Coefficients) —
If f *is a* T*-periodic function, its Fourier coefficients are:*

$$a_0 = \frac{1}{T}\int_0^T f(t)dt$$
$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)dt$$
$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)dt$$

Theorem 3.2.8 (Dirichlet Conditions) —
If f *is piecewise smooth and periodic, then its Fourier series converges to:*

$$\frac{1}{2}[f(t^+) + f(t^-)]$$

At points of continuity, it converges to $f(t)$.

Theorem 3.2.9 (Parseval's Identity) —
For a square-integrable periodic function:

$$\frac{1}{T}\int_0^T |f(t)|^2dt = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Example 3.2.5 (Square Wave) —

For the square wave $f(t) = \begin{cases} 1 & 0 \leq t < \pi \\ -1 & \pi \leq t < 2\pi \end{cases}$ with period 2π:

$$a_0 = 0$$
$$a_n = 0 \quad \text{(since } f \text{ is odd)}$$
$$b_n = \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

So the Fourier series is: $\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)t)}{2k+1}$