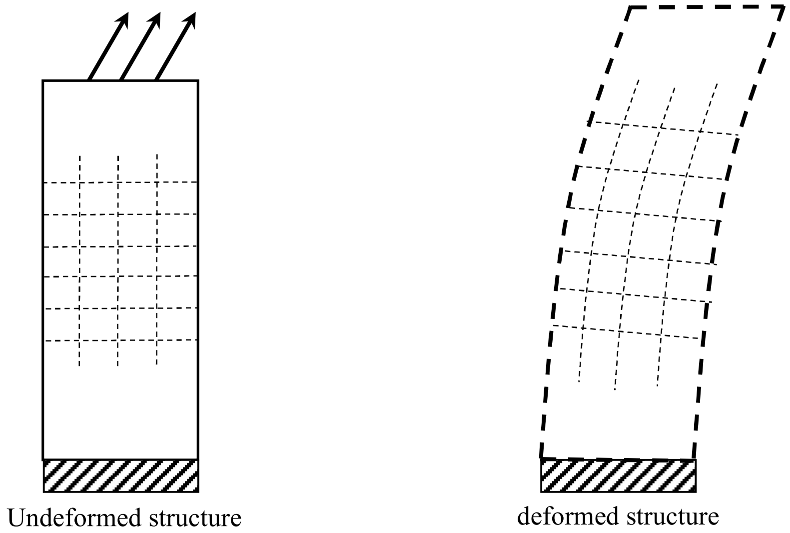
Chapter II

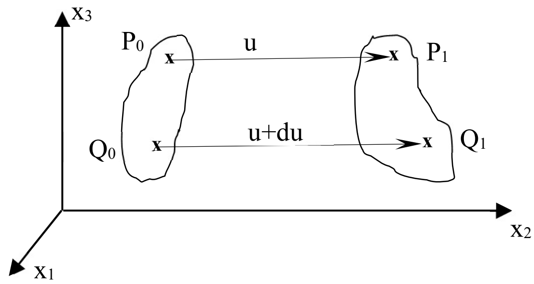
Strain Tensor

II-1- Displacement vector



Deformation of an elastic solid

In the case of linear elasticity, the assumption of small deformations is considered. Thus, we identify two neighbouring material points P_o and Q_o.



Let $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ be the displacement vector of point P

$u + du = \begin{pmatrix} u_1 + du_1 \\ u_2 + du_2 \\ u_3 + du_3 \end{pmatrix}$ the displacement vector

of point Q_o neighbouring P_o.

$$Q_0Q_1 = \begin{cases} u_1 + du_1 = u_1 + \dfrac{\partial u_1}{\partial x_1}dx_1 + \dfrac{\partial u_1}{\partial x_2}dx_2 + \dfrac{\partial u_1}{\partial x_3}dx_3 \\ u_2 + du_2 = u_2 + \dfrac{\partial u_2}{\partial x_1}dx_1 + \dfrac{\partial u_2}{\partial x_2}dx_2 + \dfrac{\partial u_2}{\partial x_3}dx_3 \\ u_3 + du_3 = u_3 + \dfrac{\partial u_3}{\partial x_1}dx_1 + \dfrac{\partial u_3}{\partial x_2}dx_2 + \dfrac{\partial u_3}{\partial x_3}dx_3 \end{cases}$$

$$Q_0Q_1 = \begin{pmatrix} u_1 + du_1 \\ u_2 + du_2 \\ u_3 + du_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$$

$$Q_0Q_1 = u + du = u + \mathrm{grad}(u)\cdot dx$$

$$Q_0Q_1 = P_0 P_1 + \mathrm{grad}(P_0 P_1)\cdot(P_0 Q_0)$$

grad(u) : displacement gradient tensor

II-2- Strain tensor

$$grad(P_0P_1) = grad\vec{u} = E + \Omega \quad \text{avec} \quad \begin{cases} \mathrm{E} = \mathrm{Sym}(grad\vec{u}) = \frac{1}{2}(grad\vec{u} + grad^t\vec{u}) \\ \Omega = \mathrm{Antisym}(grad\vec{u}) = \frac{1}{2}(grad\vec{u} - grad^t\vec{u}) \end{cases}$$

$Q_0Q_1 = P_0P_1 + \mathscr{E}\cdot(P_0Q_0) + \Omega\cdot(P_0Q_0)$ or $u+du = u + \mathscr{E}\cdot dx + \Omega\cdot dx$

Consequently, an elementary domain surrounding a point P undergoes the sum of three transformations:

- a translation of direction vector $P_0P_1 = u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$
- a rotation around point P_0 by an angle ω dual vector of an antisymmetric second-order tensor Ω (rotation tensor) such that :

$$\Omega = \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) 0 & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) & 0 \end{pmatrix} \qquad \omega = \begin{pmatrix} \omega_{23} \\ \omega_{13} \\ \omega_{12} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) \end{pmatrix}$$

$$\Omega = (*\omega) = \begin{pmatrix} 0 & -\omega_{12} & \omega_{13} \\ \omega_{12} & 0 & -\omega_{23} \\ -\omega_{13} & \omega_{23} & 0 \end{pmatrix} \qquad \text{with} \quad \omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

- pure strain (deformation) defined by the following tensor :
-
- $$\mathscr{E}=\begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

$$\mathscr{E}=\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix} \qquad \text{with} \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$$

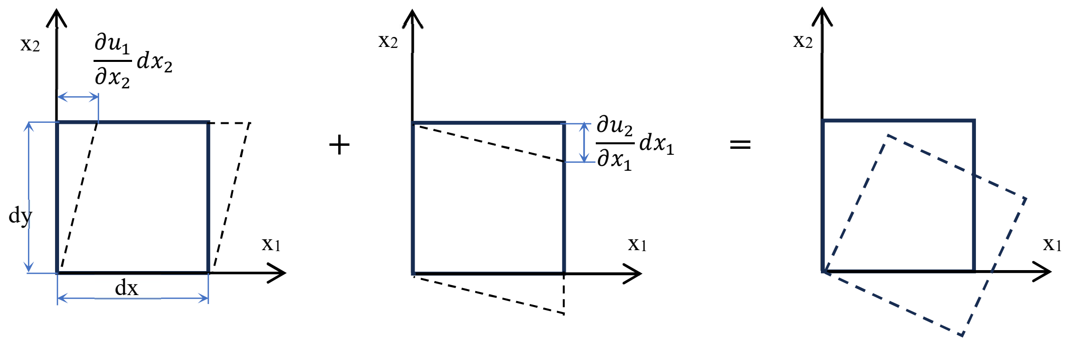
<u>two-dimensional case study :</u>

we consider the case of the two-dimensional deformation of a rectangular element of original dimensions dx_1 by dx_2.

$$\Omega = \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}\right) & 0 \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathscr{E}=\begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & 0 \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

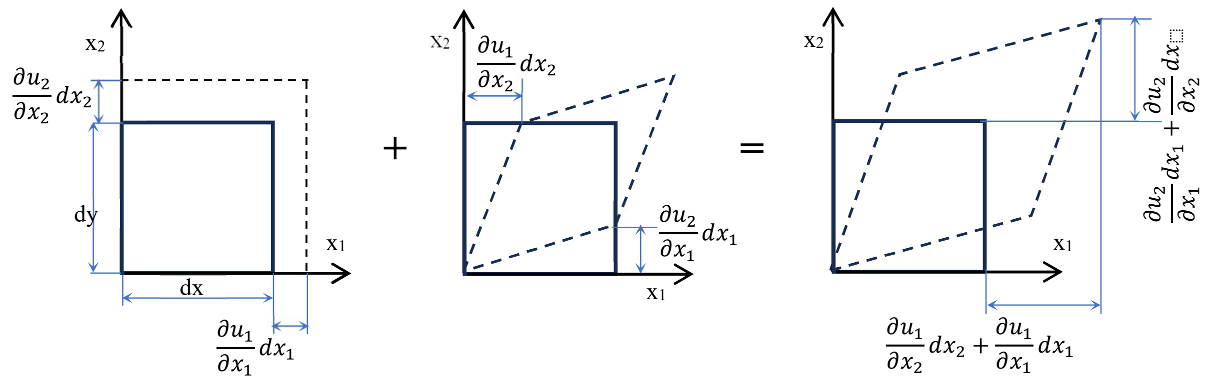
if $\varepsilon_{ij}=0 \Rightarrow \frac{\partial u_1}{\partial x_2} = -\frac{\partial u_2}{\partial x_1} \Rightarrow$

$$\Omega = \begin{pmatrix} 0 & \frac{\partial u_1}{\partial x_2} & 0 \\ \frac{\partial u_2}{\partial x_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial u_1}{\partial x_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \frac{\partial u_2}{\partial x_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



if $\omega_{ij}=0 \Rightarrow \frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1} \Rightarrow$

$$\mathscr{E}=\begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & 0 \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial u_2}{\partial x_2} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{\partial u_1}{\partial x_2} & 0 \\ \frac{\partial u_2}{\partial x_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



Case of cylindrical coordinates:

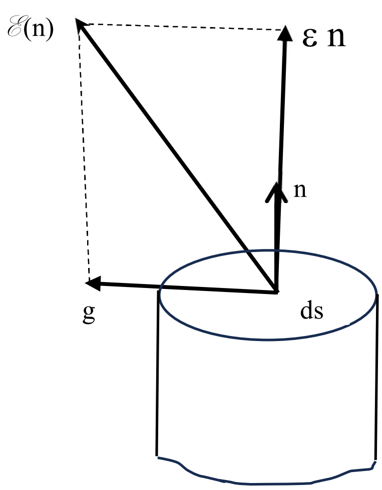
$$
\begin{aligned}
x_1 &= r\cos(\theta)\\
x_2 &= r\sin(\theta) \qquad \text{et} \qquad u=\begin{cases} u_1 = u_1(r,\theta,x_3)\\ u_2 = u_2(r,\theta,x_3)\\ u_3 = u_3(r,\theta,x_3)\end{cases}\\
x_3 &= x_3
\end{aligned}
$$

$$
grad\vec{u}=\begin{pmatrix}
\frac{\partial u_1}{\partial r} & \frac{1}{r}\left(\frac{\partial u_1}{\partial \theta}-u_2\right) & \frac{\partial u_1}{\partial x_3}\\
\frac{\partial u_2}{\partial r} & \frac{1}{r}\left(\frac{\partial u_2}{\partial \theta}+u_1\right) & \frac{\partial u_2}{\partial x_3}\\
\frac{\partial u_3}{\partial r} & \frac{1}{r}\frac{\partial u_3}{\partial \theta} & \frac{\partial u_3}{\partial x_3}
\end{pmatrix}
\qquad
\mathscr{E}=\begin{pmatrix}
\frac{\partial u_1}{\partial r} & \frac{1}{2}\left(\frac{\partial u_2}{\partial r}+\frac{1}{r}\frac{\partial u_1}{\partial \theta}-\frac{u_2}{r}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial r}\right)\\
\varepsilon_{r\theta} & \frac{1}{r}\left(\frac{\partial u_2}{\partial \theta}+u_1\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial \theta}\right)\\
\varepsilon_{r3} & \varepsilon_{\theta 3} & \frac{\partial u_3}{\partial x_3}
\end{pmatrix}
$$

$$
\Omega=\begin{pmatrix}
0 & \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_1}{\partial \theta}-\frac{u_2}{r}-\frac{\partial u_2}{\partial r}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial r}\right)\\
\frac{1}{2}\left(\frac{\partial u_2}{\partial r}-\frac{1}{r}\frac{\partial u_1}{\partial \theta}+\frac{u_2}{r}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}-\frac{\partial u_3}{\partial \theta}\right)\\
\frac{1}{2}\left(\frac{\partial u_3}{\partial r}-\frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial \theta}-\frac{\partial u_2}{\partial x_3}\right) & 0
\end{pmatrix}
$$

II-3- decomposition of the deformation

$\vec{n}$ vector normal to the surface ds



The strain vector is defined by : :

$\mathscr{E}(\vec{n}) = \varepsilon\,.\vec{n}+\vec{g}$ se compose en :

Normal deformation (elongation) along the vector n :

$$\varepsilon = \vec{n}^{\,t}\,\mathscr{E}\,\vec{n}$$

Shear deformation (slip) on the plane of normal n :

$$g = \left|(*\vec{n})\mathscr{E}\,\vec{n}\right| = \sqrt{\left[\mathscr{E}(\vec{n})\right]^2-\varepsilon^2}$$

II-4- Principal Strains and principal directions

The tensor $\mathscr{E}$ being symmetrical, therefore there exists a basis {X_1, X_2, X_3} called principal basis on which the matrix ($\mathscr{E}$) is diagonal:

$$\mathscr{E} = \begin{pmatrix} \varepsilon_{\mathrm{I}} & 0 & 0 \\ 0 & \varepsilon_{\mathrm{II}} & 0 \\ 0 & 0 & \varepsilon_{\mathrm{III}} \end{pmatrix}$$

The principal strains (or elongations) ε_I, ε_{II} and ε_{III} represent the eigenvalues of the matrix ($\mathscr{E}$) which can be obtained by solving the following equation:

$$\det(\mathscr{E} - \lambda \mathrm{I}) = 0$$

Expanding the determinant produces a cubic equation in terms of λ *(called characteristic equation)*:

$$det[A - \lambda I] = -\lambda^3 + I_a\lambda^2 - I_B\lambda + I_C = 0$$

Where the scalars I_a, I_b, and I_c are called the *fundamental invariants of the tensor* $\mathscr{E}$ such as :

$$I_a = tr(A) = A_{ii} = A_{11} + A_{22} + A_{33} = \lambda_1 + \lambda_2 + \lambda_3$$

$$I_b = \tfrac{1}{2}(A_{ii}A_{jj} - A_{ij}A_{ij}) = \begin{vmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{23} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{13} & A_{33} \end{vmatrix} =$$

$$= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$$

$$I_c = \det(A) = det\left[A_{ij}\right] = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$$

The roots of the characteristic equation determine the allowable values for λ ($\lambda_1, \lambda_2, \lambda_3$) such as :

$\varepsilon_I=\lambda_1$ $\varepsilon_{II}=\lambda_2$ $\varepsilon_{III}=\lambda_3$

Each of the values of λ (or ε_i) can be substituted into the following vector relation to find the principal directions X_1 , X_2 et X_3 :

$$(\mathscr{E} - \varepsilon_{\mathrm{i}}\mathrm{I})\,\vec{\mathrm{X}}_{\mathrm{i}} = \vec{0}$$

The unit vectors X_1, X_2 and X_3 of the principal basis verify the following vector relations:

$$\begin{cases} \mathrm{X}_1 = \mathrm{X}_2 \wedge \mathrm{X}_3 \\ \mathrm{X}_2 = \mathrm{X}_3 \wedge \mathrm{X}_1 \\ \mathrm{X}_3 = \mathrm{X}_1 \wedge \mathrm{X}_2 \end{cases}$$

III-5- Spherical and Deviatoric Strains

In particular applications it is convenient to decompose the strain tensor into two parts called spherical and deviatoric strain tensors.

$$\mathscr{E} = \mathscr{E}_{\mathrm{s}} + \mathscr{E}_{\mathrm{d}}$$

The spherical strain tensor is defined by

$$\mathscr{E}_s = \frac{e}{3} I \qquad (e = tr(\mathscr{E}))$$

The deviatoric strain tensor is defined by

$$\mathscr{E}_d = \mathscr{E} - \frac{e}{3} I$$

The spherical strain represents only volumetric deformation and is an isotropic tensor, being the same in all coordinate systems. The deviatoric strain tensor then accounts for changes in shape of material elements. It can be shown that the principal directions of the deviatoric strain are the same as those of the strain tensor.