Chapter 3

Differential Calculus

Differentiation and the rules of differentiation are fundamental concepts in calculus in mathematics. Differentiation is concerned with the instantaneous rate of change of a given function, while the rules of differentiation form a set of rules and principles that facilitate the calculation of derivatives in specific ways and provide us with information about the properties of derivative functions.

3.1 Derivative and derivation laws

3.1.1 Derivative at a Point

The derivative of a function at a point captures the instantaneous rate of change of the function at that specific location. This fundamental concept in calculus has applications across physics, engineering, economics, and many other fields.

Definition 3.1 (Differentiability at a Point)

Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ be a function. For $x_0 \in I$, we say f is **differentiable** at x_0 if the following limit exists:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

When this limit exists, it is called the **derivative** of f at x_0 and denoted by $f'(x_0)$ or $\frac{df}{dx}\big|_{x=x_0}$.

Example 3.1 (Linear Function)

For $f(x) = 3x + 2$, the derivative at any point x_0 is:

$$\begin{aligned} f'(x_0) &= \lim_{h\to 0} \frac{[3(x_0+h)+2]-[3x_0+2]}{h} \\ &= \lim_{h\to 0} \frac{3h}{h} = 3 \end{aligned}$$

This shows the derivative (slope) is constant, as expected for a linear function.

Example 3.2 (Quadratic Function)

For $f(x) = x^2$, the derivative at $x_0 = 1$ is:

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{(1+h)^2 - 1^2}{h} \\ &= \lim_{h\to 0} \frac{1+2h+h^2-1}{h} \\ &= \lim_{h\to 0}(2+h) = 2 \end{aligned}$$

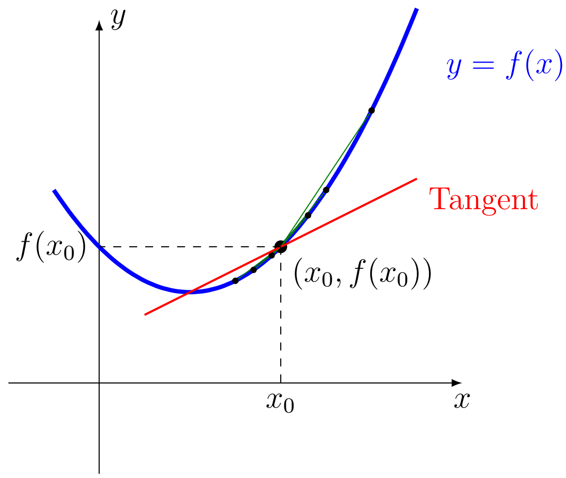
This matches our general formula that $\frac{d}{dx}x^2 = 2x$.

Definition 3.2 (Differentiability on an Interval)

A function f is **differentiable on an open interval** I if it is differentiable at every point $x \in I$. The function $f' : I \to \mathbb{R}$ that maps each x to its derivative $f'(x)$ is called the **derivative function**.

3.1.2 Geometric Interpretation of the Derivative

The derivative has a fundamental geometric interpretation as the slope of the tangent line to the function's graph.



The tangent line at $(x_0, f(x_0))$ has equation:

$$y = f'(x_0)(x - x_0) + f(x_0)$$

Theorem 3.1 (Differentiability Implies Continuity)

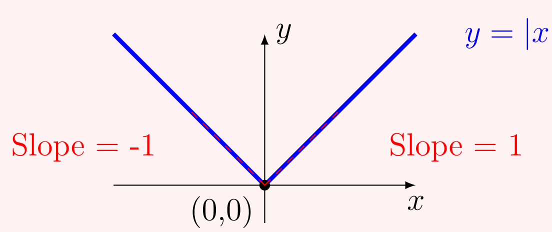
If f is differentiable at x_0, then f is continuous at x_0. The converse is not true - continuity does not imply differentiability.

Example 3.3 (Non-Differentiable Function)

The absolute value function $f(x) = |x|$ is continuous at $x = 0$ but not differentiable there:

$$\lim_{h \to 0^+} \frac{|0 + h| - |0|}{h} = 1$$
$$\lim_{h \to 0^-} \frac{|0 + h| - |0|}{h} = -1$$

The left and right limits disagree, so the derivative doesn't exist.



Indeed, the rate of increase at $x_0 = 0$ achieves:

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x} = \begin{cases} +1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

3.1.3 Rules of Differentiation

Theorem 3.2 (Basic Differentiation Rules)

Let f, g be differentiable functions and $c \in \mathbb{R}$ a constant:

- **Linearity**:

$$(f+g)' = f' + g'$$
$$(cf)' = cf'$$

- **Product Rule**:

$$(fg)' = f'g + fg'$$

- **Quotient Rule**:

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \quad (g \neq 0)$$

- **Chain Rule**:

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Example 3.4 (Product Rule Application)

Let $f(x) = x^2 \sin(x)$. Then:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^2) \cdot \sin(x) + x^2 \cdot \frac{d}{dx}(\sin(x)) \\ &= 2x \sin(x) + x^2 \cos(x) \end{aligned}$$

Example 3.5 (Chain Rule Application)

For $f(x) = \sqrt{x^2+1}$:

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x^2+1}} \cdot \frac{d}{dx}(x^2+1) \\ &= \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}} \end{aligned}$$

3.1.4 Derivatives of Elementary Functions

Function	**Derivative**
c (constant)	0
x^n	nx^{n-1}
e^x	e^x
a^x	$a^x \ln a$
$\ln x$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{x \ln a}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$sech^2 x$

Example 3.6 (Higher Order Derivatives)

For $f(x) = x^4 + 3x^2 - 2x$:

$$\begin{aligned} f'(x) &= 4x^3 + 6x - 2 \\ f''(x) &= 12x^2 + 6 \\ f'''(x) &= 24x \\ f^{(4)}(x) &= 24 \\ f^{(n)}(x) &= 0 \quad \text{for } n \geq 5 \end{aligned}$$

Theorem 3.3 (Differentiation of Inverse Functions)

If f is bijective and differentiable at x_0 with $f'(x_0) \neq 0$, then its inverse f^{-1} is differentiable at $y_0 = f(x_0)$ and:

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

Example 3.7 (Derivative of Inverse)

For $f(x) = e^x$, its inverse is $f^{-1}(x) = \ln x$. Then:

$$(\ln x)' = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

3.2 Successive Derivatives

Successive derivatives, also known as higher-order derivatives, extend the concept of differentiation to multiple levels. These derivatives have important applications in Taylor series, differential equations, and the study of function behavior.

3.2.1 Definition and Notation

Definition 3.3 (Higher-Order Derivatives)

Let $f : I \to \mathbb{R}$ be a differentiable function. We define the successive derivatives recursively:

- $f^{(0)} = f$ (the zeroth derivative is the function itself)
- $f^{(1)} = f'$ (the first derivative)
- $f^{(2)} = f'' = (f')'$ (the second derivative)
- $f^{(n+1)} = \left(f^{(n)}\right)'$ for $n \geq 0$ (the $(n+1)$-th derivative)

If $f^{(n)}$ exists, we say f is **n-times differentiable**. If all orders exist, f is **infinitely differentiable** or **smooth**.

Example 3.8 (Polynomial Function)

For $f(x) = x^5 - 3x^2 + 2$, we have:

$$f'(x) = 5x^4 - 6x$$
$$f''(x) = 20x^3 - 6$$
$$f'''(x) = 60x^2$$
$$f^{(4)}(x) = 120x$$
$$f^{(5)}(x) = 120$$
$$f^{(n)}(x) = 0 \quad \text{for } n \geq 6$$

This shows that polynomials have finite differentiability orders.

Example 3.9 (Exponential Function)

For $f(x) = e^{2x}$, all derivatives exist:

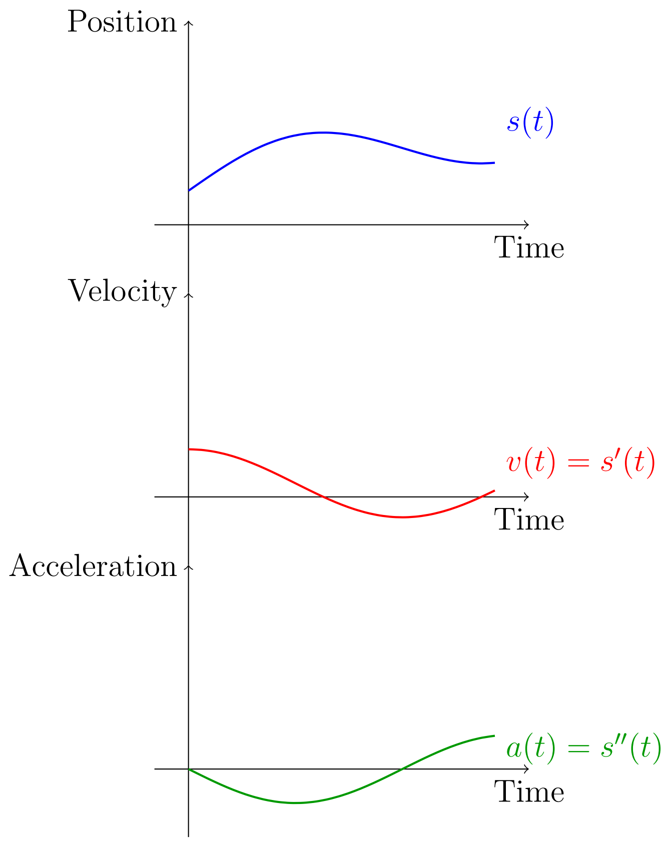
$$f'(x) = 2e^{2x}$$
$$f''(x) = 4e^{2x}$$
$$f'''(x) = 8e^{2x}$$
$$\vdots$$
$$f^{(n)}(x) = 2^n e^{2x}$$

The exponential function is infinitely differentiable.

Physical Interpretations

Higher-order derivatives have important physical meanings:

- **First derivative**: Velocity (rate of change of position)
- **Second derivative**: Acceleration (rate of change of velocity)
- **Third derivative**: Jerk (rate of change of acceleration)
- **Fourth derivative**: Jounce or snap



Leibniz's Rule for Higher Derivatives

The following theorem generalizes the product rule to higher derivatives:

Theorem 3.4 (Leibniz's Rule)

Let $f, g : I \to \mathbb{R}$ be n-times differentiable functions. Then:

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

where $\binom{n}{k}$ is the binomial coefficient and $f^{(0)} = f$.

Proof. We proceed by mathematical induction.

Base case ($n = 0$):

$$(f \cdot g)^{(0)} = f \cdot g = \sum_{k=0}^{0} \binom{0}{k} f^{(k)} g^{(0-k)}$$

Inductive step: Assume the formula holds for n, then for $n+1$:

$$\begin{aligned}
(f \cdot g)^{(n+1)} &= \frac{d}{dx}\left((f \cdot g)^{(n)}\right) \\
&= \frac{d}{dx}\left(\sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(f^{(k+1)} g^{(n-k)} + f^{(k)} g^{(n+1-k)}\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} f^{(k+1)} g^{(n-k)} + \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n+1-k)}
\end{aligned}$$

Let $p = k+1$ in the first sum:

$$\begin{aligned}
&= \sum_{p=1}^{n+1} \binom{n}{p-1} f^{(p)} g^{(n+1-p)} + \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n+1-k)} \\
&= \sum_{k=1}^{n+1} \binom{n}{k-1} f^{(k)} g^{(n+1-k)} + \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n+1-k)}
\end{aligned}$$

Combine the sums and use Pascal's identity $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$:

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)} g^{(n+1-k)}$$

This completes the induction. ■

Example 3.10 (Application of Leibniz's Rule)

Compute the 4th derivative of $f(x) = x^2 e^x$.

Let $g(x) = x^2$ and $h(x) = e^x$. Then:

$$\begin{aligned}
(g \cdot h)^{(4)} &= \sum_{k=0}^{4} \binom{4}{k} g^{(k)} h^{(4-k)} \\
&= \binom{4}{0} x^2 e^x + \binom{4}{1} (2x) e^x + \binom{4}{2} (2) e^x \\
&= x^2 e^x + 8x e^x + 12 e^x \\
&= e^x (x^2 + 8x + 12)
\end{aligned}$$

Note that $g^{(k)} = 0$ for $k \geq 3$.

Important Properties

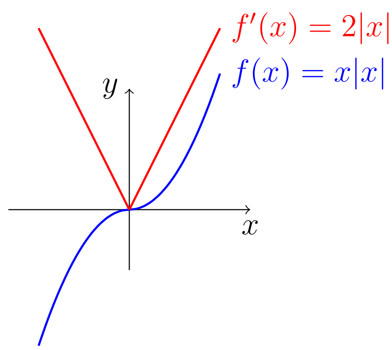
Proposition 3.2.1 (Linearity of Higher Derivatives)**.** *For n-times differentiable functions f, g and constants a, b:*

$$(af + bg)^{(n)} = af^{(n)} + bg^{(n)}$$

Proposition 3.2.2 (Derivative of Composition)**.** *For f and g sufficiently differentiable, the chain rule extends to higher derivatives, though the general formula (Faà di Bruno's formula) is more complex than Leibniz's rule.*

Example 3.11 (Non-Example: Differentiability)

The function $f(x) = x|x|$ is differentiable everywhere, but its derivative $f'(x) = 2|x|$ is not differentiable at $x = 0$. Thus, f is once but not twice differentiable at 0.

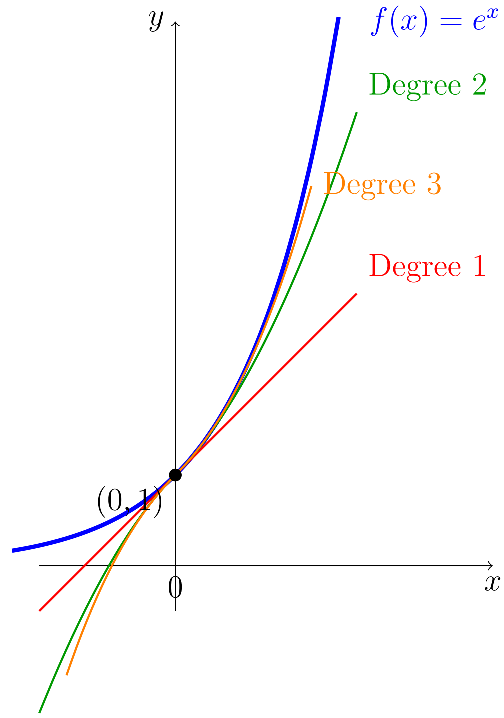


3.3 Limited Expansions and Taylor Approximations

Limited expansions provide powerful tools for approximating functions near specific points using polynomials. These approximations become increasingly accurate as we include more terms in the expansion.

3.3.1 Introduction to Limited Expansions

The basic idea is to approximate a function $f(x)$ near a point x_0 using a polynomial that matches the function's value and derivatives at x_0.



3.3.2 Taylor's Theorem

Theorem 3.5 (Taylor's Theorem with Peano Remainder)

Let $f : I \to \mathbb{R}$ be n-times differentiable at $x_0 \in I$. Then there exists a function ε such that:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + (x - x_0)^n \varepsilon(x - x_0)$$

where $\lim_{x \to x_0} \varepsilon(x - x_0) = 0$.

Definition 3.4 (Taylor Polynomial)

The **Taylor polynomial** of degree n for f at x_0 is:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

The remainder term is $R_n(x) = f(x) - P_n(x)$.

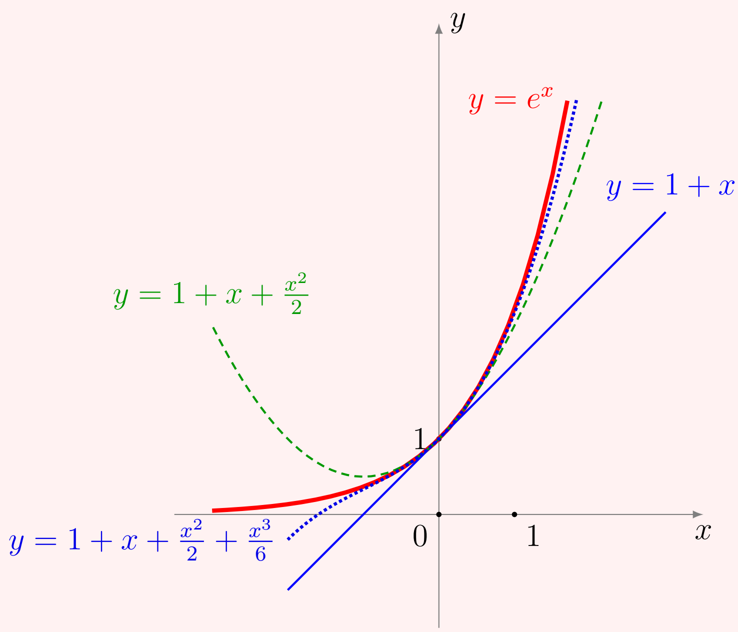
Example 3.12 (Exponential Function)

For $f(x) = e^x$ at $x_0 = 0$:

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= 1 \\ f''(0) &= 1 \\ &\vdots \\ f^{(n)}(0) &= 1 \end{aligned}$$

Thus the Taylor polynomial is:

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$



3.3.3 Maclaurin Series (Special Case at $x_0 = 0$)

When the expansion point is $x_0 = 0$, the Taylor series is called a Maclaurin series.

Theorem 3.6 (Maclaurin Series)

For f infinitely differentiable near 0:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

when the series converges.

Common Maclaurin Expansions

Function	**Maclaurin Expansion**
e^x	$\sum_{k=0}^{\infty} \frac{x^k}{k!}$
$\sin x$	$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$
$\cos x$	$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$
$\ln(1+x)$	$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$
$(1+x)^\alpha$	$\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$
$\frac{1}{1-x}$	$\sum_{k=0}^{\infty} x^k$

Example 3.13 (Sine and Cosine Functions)

For $f(x) = \sin x$:

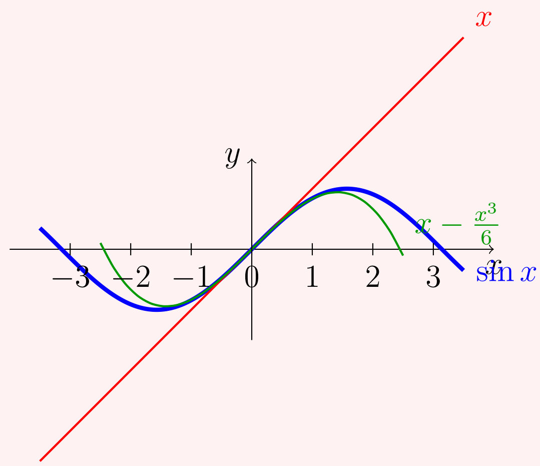
$$f(0) = 0$$
$$f'(0) = 1$$
$$f''(0) = 0$$
$$f'''(0) = -1$$
$$f^{(4)}(0) = 0$$
$$\vdots$$

Thus:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Similarly for $\cos x$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$



3.3.4 Operations on Limited Expansions

When working with limited expansions, we can perform various operations while maintaining the approximation order.

Proposition 3.3.1 (Algebraic Operations)**.** *Let* $f(x) = P_n(x) + o(x^n)$ *and* $g(x) = Q_n(x) + o(x^n)$ *be limited expansions at* 0*.*

- ***Sum***: $(f + g)(x) = P_n(x) + Q_n(x) + o(x^n)$
- ***Product***: $(f \cdot g)(x) = P_n(x)Q_n(x) + o(x^n)$ *(truncate to degree n)*
- ***Composition***: *If* $g(0) = 0$*, then* $(f \circ g)(x) = P_n(Q_n(x)) + o(x^n)$
- ***Division***: *For* $g(0) \neq 0$, $\frac{1}{g(x)} = \frac{1}{Q_n(x)} + o(x^n)$

Example 3.14 (Product of Expansions)

Compute the limited expansion of $e^x \cos x$ at order 3:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3) \\ \cos x &= 1 - \frac{x^2}{2} + o(x^3) \\ e^x \cos x &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\left(1 - \frac{x^2}{2}\right) + o(x^3) \\ &= 1 + x + \left(\frac{1}{2} - \frac{1}{2}\right)x^2 + \left(\frac{1}{6} - \frac{1}{2}\right)x^3 + o(x^3) \\ &= 1 + x - \frac{x^3}{3} + o(x^3) \end{aligned}$$

3.3.5 Applications and Examples

Example 3.15 (Arctangent Function)

Compute the limited expansion of $\arctan x$ at order 5:

$$\begin{aligned} \frac{d}{dx}\arctan x &= \frac{1}{1+x^2} = 1 - x^2 + x^4 + o(x^5) \\ \arctan x &= \int (1 - x^2 + x^4)dx + o(x^6) \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^6) \end{aligned}$$

Example 3.16 (Tangent Function)

Compute $\tan x$ at order 5 using $\tan x = \frac{\sin x}{\cos x}$:

$$\begin{aligned} \sin x &= x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^6) \\ \cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5) \\ \frac{1}{\cos x} &= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + o(x^5) \\ \tan x &= \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\left(1 + \frac{x^2}{2} + \frac{5x^4}{24}\right) + o(x^5) \\ &= x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^5) \end{aligned}$$

Example 3.17 (Composite Function)

Compute $\sin(\ln(1+x))$ at order 3:

$$\begin{aligned} \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) \\ \sin u &= u - \frac{u^3}{6} + o(u^3) \\ \sin(\ln(1+x)) &= \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) - \frac{x^3}{6} + o(x^3) \\ &= x - \frac{x^2}{2} + \frac{x^3}{6} + o(x^3) \end{aligned}$$

3.3.6 Error Estimation

Theorem 3.7 (Taylor's Remainder Theorem)

If f is $(n+1)$-times differentiable on an interval containing x_0, then for each x in the interval, there exists c between x_0 and x such that:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

Example 3.18 (Error Bound for e^x)

Approximate $e^{0.1}$ using a 3rd degree Taylor polynomial:

$$e^{0.1} \approx 1 + 0.1 + \frac{0.1^2}{2} + \frac{0.1^3}{6} \approx 1.1051667$$

The error satisfies:

$$|R_3(0.1)| \leq \frac{e^{0.1}}{24}(0.1)^4 \leq \frac{1.1052}{24} \times 10^{-4} \approx 4.6 \times 10^{-6}$$

The actual value is $e^{0.1} \approx 1.1051709$, with error $\approx 4.2 \times 10^{-6}$.