TP #3: Numerical Methods for Finding Roots: Fixed-Point and Newton–Raphson

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1. Fixed-Point Iteration Method

1.1 Principle

Consider the nonlinear equation:

$$f(x) = 0$$

We rewrite it in the form:

$$x = g(x)$$

and define the iteration:

$$x_{n+1} = g(x_n)$$

The method converges to the true root x^* if the condition $|g'(x^*)| < 1$ is satisfied.

1.2 Numerical Example

We wish to find the root of:

$$f(x) = \cos(x) - x = 0$$

which can be expressed as:

$$x = \cos(x)$$

Let $g(x) = \cos(x)$ and start from $x_0 = 0.5$.

Iteration (n)	x_n	$x_{n+1} = g(x_n)$	$\lvert x_{n+1} - x_n \rvert$
0	0.5000	0.8776	0.3776
1	0.8776	0.6390	0.2386
2	0.6390	0.8027	0.1637
3	0.8027	0.6948	0.1079
4	0.6948	0.7682	0.0734
5	0.7682	0.7192	0.0490
6	0.7192	0.7524	0.0332
7	0.7524	0.7301	0.0223
8	0.7301	0.7451	0.0150

The root converges slowly to approximately:

$$x^* \approx 0.7391$$

1.3 MATLAB Code

Listing 1: Fixed-Point Iteration Method in MATLAB

```
% Fixed-Point Iteration Method
clc; clear; close all;

f = @(x) cos(x) - x ; g = @(x) cos(x) ;

x0 = 0.5 ; tol = 1e-6 ; maxIter = 100 ; iter = 0 ;

while iter < maxIter
    x1 = g(x0);
    err = abs(x1 - x0);
     if err < tol
        xroot = x1 ;
        break ;
    end
    x0 = x1 ; iter = iter + 1 ;
end
```

2. Newton–Raphson Method

2.1 Principle

For a differentiable function $f(x)$, we use:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This method converges **quadratically** if $f'(x^*) \neq 0$ and x_0 is sufficiently close to the true root.

2.2 Numerical Example

Let:

$$f(x) = \cos(x) - x$$

then

$$f'(x) = -\sin(x) - 1$$

Starting from $x_0 = 0.5$:

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}	Error
0	0.5000	0.3776	-1.4794	0.7552	0.2552
1	0.7552	-0.0271	-1.6858	0.7391	0.0161
2	0.7391	-0.0004	-1.6736	0.7391	0.0000

The method rapidly converges to the same root:

$$x^* \approx 0.7391$$

2.3 MATLAB Code

Listing 2: Newton–Raphson Method in MATLAB

```
% Newton-Raphson Method
clc; clear; close all;

f = @(x) cos(x) - x ; df = @(x) -sin(x) - 1 ;

x0 = 0.5 ; tol = 1e-6 ; maxIter = 100 ; iter = 0 ;

while iter < maxIter
    x1 = x0 - f(x0)/df(x0);
    err = abs(x1 - x0);

    if err < tol
       xroot = x1 ;
       break ;
    end
    x0 = x1;
    iter = iter + 1;
end
```

3. Comparison and Remarks

Method	Convergence Order	Iterations Needed	Root Approximation
Fixed-Point Iteration	Linear	$\approx$ 8–10	0.7391
Newton–Raphson	Quadratic	$\approx$ 2–3	0.7391

Remarks:

- The fixed-point method is simple but can be slow and may not converge if $|g'(x)| \geq 1$.
- The Newton–Raphson method converges much faster but requires computation of the derivative $f'(x)$.
- Both methods give the same root, but the rate of convergence is very different.