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COURSE TITLE

Mathematic and statistic Level 1^{st} year LMD

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General introduction

Mathematics and statistics are important for biology students because they provide the tools to analyze data, build models, and conduct research. These subjects are crucial for understanding biological concepts quantitatively, whether it's tracking population changes, analyzing genetic data, or understanding rates of chemical reactions.

Key areas where math and statistics are applied in biology

- **Data analysis**: Statistics are used to describe data, uncover patterns, and determine if results are statistically significant.
- **Modeling**: Mathematics is used to create models that can predict or describe biological phenomena, such as how a disease spreads or how populations change over time.
- **Experimental design**: Statistics is essential for designing experiments that will yield reliable results and avoid common pitfalls in data collection.
- **Specific disciplines**:
  - Biochemistry: Mathematical computer models help in understanding complex reactions and processes.
  - Genetics: Math is used in computer programs for analyzing DNA sequences.
  - Zoology: Mathematical tools can express the relatedness of species and estimate when they diverged from a common ancestor.
  - Medicine: Diagnostic tools like MRI and EEG rely on mathematical principles.
- **Lab work**: Basic math is necessary for practical tasks like preparing solutions with correct concentrations, a fundamental skill in many biology labs.

Chapter 1

Limits and Continuity

Introduction

In this chapter, we study two fundamental notions of calculus: **limits** and **continuity**. These concepts are essential for understanding how functions behave and change, which is useful in many areas of biology such as population dynamics, enzyme kinetics, and growth models.

1.1 What is a function?

A function is a relation that associates each element of a set called the starting set with an element of another set called the ending set.

$$f : D \to A$$
$$x \to f(x).$$

Définition 1.1 *Let f be a function on D_f.*

- *The function f is* ***odd*** *iff (if and only if) the following statements are correct.*
  1. *$\forall x \in D_f$ then $-x \in D_f$*
  2. *$\forall x \in D_f$ we have $f(-x) = -f(x)$.*
- *The function f is* ***even*** *iff the following statements are correct.*
  1. *$\forall x \in D_f$ then $-x \in D_f$*
  2. *$\forall x \in D_f$ we have $f(-x) = f(x)$.*
- *A* ***periodic*** *function is a function that repeats itself in regular intervals or periods. The function f is said to be periodic if $\forall\ x \in D_f\ \exists\ p \in \mathbb{R}^*$:*

$$f(x + p) = f(x)$$

.

Symbols	Explanation
$\forall$	for all
$\exists$	exists
$\in$	in
$\Rightarrow$	implies
$\Leftrightarrow$	equivalane
iff	if and only if

1.2 Limit of a Function

The **limit** of a function describes the value that the function approaches as the variable approaches a given number.

2.2 Formal Definition

Let $f(x)$ be defined around a. We say that $f(x)$ tends to a limit L as x tends to a, and we write:

$$\lim_{x \to a} f(x) = L$$

if for every $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$|x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Left and Right Limits

- The **right limit** of f at a is $\lim\limits_{x \to a^+} f(x)$ (when $x > a$). - The **left limit** of f at a is $\lim\limits_{x \to a^-} f(x)$ (when $x < a$).

If both limits exist and are equal, then the limit at a exists.

2.4 Example

Let

$$f(x) = \begin{cases} x^2, & x < 2, \\ 3x - 2, & x \geq 2. \end{cases}$$

We have:

$$\lim_{x \to 2^-} f(x) = (2)^2 = 4, \quad \lim_{x \to 2^+} f(x) = 3(2) - 2 = 4$$

Therefore, $\lim_{x \to 2} f(x) = 4$.

1.3 Continuous Functions

Definition

A function f is said to be **continuous** at a point a if:

$$\lim_{x \to a} f(x) = f(a)$$

That is, the limit of $f(x)$ as x approaches a is equal to the actual value of f at that point.

Continuity on an Interval

A function f is continuous on an interval I if it is continuous at every point of I.

Examples

1. $f(x) = x^2$ is continuous everywhere on $\mathbb{R}$. 2. $f(x) = \frac{1}{x}$ is continuous on $\mathbb{R} \setminus \{0\}$.

Types of Discontinuity

- **Jump discontinuity:** when the left and right limits exist but are different.

$$f(x) = \begin{cases} 2x+1, & x < 1 \\ x+3, & x \geq 1 \end{cases}$$

  Then,

$$\lim_{x \to 1^-} f(x) = 3, \quad \lim_{x \to 1^+} f(x) = 4$$

  Since the two limits are not equal, f has a **jump discontinuity** at $x = 1$.

- **Infinite discontinuity:** when the function tends to infinity near a point.
  Example: $f(x) = \frac{1}{x}$ at $x = 0$.

- **Removable discontinuity:** when the limit exists but is not equal to $f(a)$.
  Example: $f(x) = \frac{x^2-1}{x-1}$ at $x = 1$.

The Intermediate Value Theorem (IVT)

Theorem Statement

If f is continuous on a closed interval $[a, b]$ and N is any number between $f(a)$ and $f(b)$, then there exists at least one $c \in [a, b]$ such that:

$$f(c) = N$$

Example

Let $f(x) = x^3 - x - 2$ on $[1, 2]$.

$$f(1) = -2, \quad f(2) = 4$$

Since 0 is between -2 and 4, by the IVT, there exists $c \in (1, 2)$ such that $f(c) = 0$. Numerically, $c \approx 1.52$.

Chapter 2

Differentiability

Introduction

Before studying differentiability, we must know the concepts of **function**, **limit**, and **continuity**. Differentiability helps us describe how fast a quantity changes, for example, how fast a population grows or how a reaction rate changes with temperature. It tells us when a function can be approximated by a straight line near a point.

2.1 Definition of Differentiability (One Variable)

Definition

Let f be a function defined in the neighborhood of x_0. We say that f is differentiable at a point x_0 if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists in $\mathbb{R}$. When this limit exists, it is denoted by $f'(x_0)$ and called the derivative of f at x_0.

Remark *If we put $x - x_0 = h$, the quantity $\frac{f(x) - f(x_0)}{x - x_0}$ becomes $\frac{f(x_0 + h) - f(x_0)}{h}$. So we can define the notion of differentiability of f at x_0 in the following way:*

$$f \text{ is differentiable at the point } x_0 \Leftrightarrow \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \text{ exists in } \mathbb{R}$$

Notations:

We can use the notations $f'(x_0)$, $Df(x_0)$, $\frac{df}{dx}(x_0)$ to designate the derivative of f at x_0.

Example

1. The function $f(x) = x^2$ is differentiable at any point $x_0 \in \mathbb{R}$ and the derivative $f'(x_0) = 2x_0$. As an explanation, given $x_0 \in \mathbb{R}$ we have:

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{(x_0+h)^2 - x_0^2}{h} = \lim_{h\to 0}(h+2x_0) = 2x_0.$$

2. The function $f(x) = \sin(x)$ is differentiable at any point $x_0 \in \mathbb{R}$ and the derivative $f'(x_0) = \cos(x_0)$. As an explanation, given $x_0 \in \mathbb{R}$ we have:

$$\begin{aligned}\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} &= \lim_{h\to 0} \frac{\sin(x_0+h)-\sin(x_0)}{h} \\ &= \lim_{h\to 0} \cos\left(\frac{2x_0+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} = \cos(x_0)\end{aligned}$$

Definition (Left and right derivative)

1. Let f be a function defined on an interval of type $[x_0, x_0+\alpha[$ with $\alpha > 0$. We say that f is right-differentiable at x_0 iff:

$$\lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h}$$

exists in $\mathbb{R}$. This limit is denoted by $f'_r(x_0)$ and is called the right derivative of f at x_0.

2. Let f be a function defined on an interval of type $]x_0-\alpha, x_0]$ with $\alpha > 0$. We say that f is left-differentiable at x_0 iff:

$$\lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h}$$

exists in $\mathbb{R}$. This limit is denoted by $f'_l(x_0)$ and is called the left derivative of f at x_0.

Proposition

Let f be a function defined in the neighborhood of x_0, we have:

$$f \text{ is differentiable at } x_0 \iff \begin{cases} f \text{ is differentiable on the right and left at } x_0 \\ \quad\text{and} \\ f'_r(x_0) = f'_l(x_0) \end{cases}$$

Example

Let $f(x) = |x|$, we have:

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} -\frac{h}{h} = -1 = f_l'(0)$$
$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1 = f_r'(0)$$

$\Longrightarrow$ The function f is differentiable on the right and on the left at $x_0 = 0$ and moreover $f_r'(0) = 1$ and

$$f_l'(0) = -1, \text{ so } f_l'(0) \neq f_r'(0) \implies f \text{ is not differentiable at } x_0 = 0$$

Geometrical interpretation

The figure below shows the graph of a function $y = f(x)$:

The ratio $\dfrac{f(x_0+h)-f(x_0)}{h} = \tan(\theta)$ is the slope of the straight line joining point $A(x_0, f(x_0))$ to point $B(x_0+h, f(x_0+h))$ on the graph. When $h \to 0$, this line tends towards the tangent (AC) to the curve at a point $A(x_0, f(x_0))$. So we get:

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \tan(\alpha) = \frac{CD}{AD}$$

is the slope of the tangent to the curve at point $A(x_0, f(x_0))$.

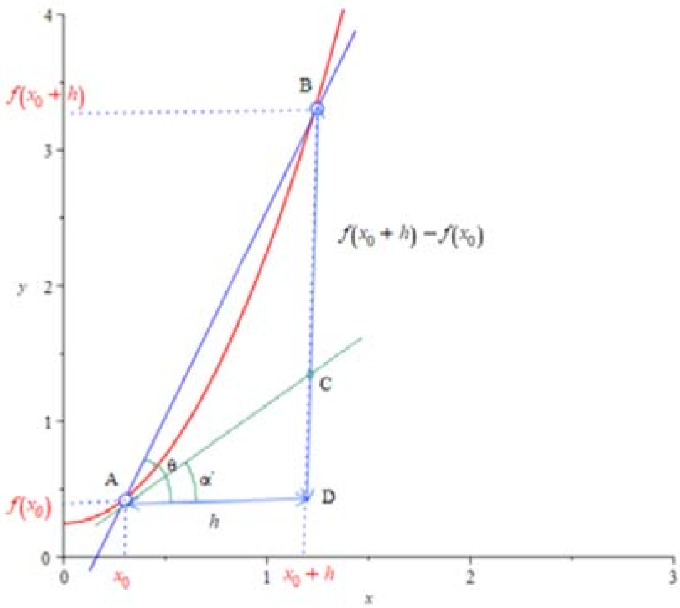

Figure : Geometrical Interpretation of Differentiability at a point x_0

Remark *According to the figure above, the equation of the tangent to the curve $y = f(x)$ at the point $A(x_0, f(x_0))$ is $y - f(x_0) = f'(x_0)(x - x_0)$*

Proposition

Let f be a function differentiable at a point x_0, then f is continuous at x_0.

Proof:

We have: $\lim_{x\to x_0}(f(x)-f(x_0)) = \lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{x-x_0}\right)(x-x_0)$

Since f is differentiable at x_0 we get:

$\lim_{x\to x_0}(f(x)-f(x_0)) = \lim_{x\to x_0} f'(x_0)(x-x_0) = 0 \implies f$ is continuous at x_0

Remark *The opposite of this theorem is incorrect. A function can be continuous at a point x_0 without being differentiable at the same point. For example, the function $x \mapsto |x|$ is continuous at $x_0 = 0$ but not differentiable at the same point.*

Differential on an interval. Derivative function.

Definition

Let f be a function defined on an open interval I. We say that f is differentiable on I if: it is differentiable at any point on I. The function defined on I by: $x \mapsto f'(x)$ is called the derivative function or simply the derivative of the function f and is denoted by f' ou $\frac{df}{dx}$.

Remark *let f be a function defined on an interval I and $a, b \in \mathbb{R} \cup \{+\infty, -\infty\}$ then:*

- *We say that f is differentiable on $I = [a, b]$ iff: it is differentiable on the open interval $]a, b[$ and differentiable on the right at a and on the left at b.*
- *We say that f is differentiable on $I = [a, b[$ if: it is differentiable on the open interval $]a, b[$ and differentiable on the right at a.*
- *We say that f is differentiable on $I =]a, b]$ if: it is differentiable on the open interval $]a, b[$ and differentiable on the left at b.*

Operations on differentiable functions

Proposition : (At a point)

Let f, g be two functions differentiable at x_0, then we have:

- $f + g$ is differentiable at x_0 et $(f+g)'(x_0) = f'(x_0) + g'(x_0)$
- $f.g$ is differentiable at x_0 et $(f.g)'(x_0) = f'(x_0).g(x_0) + f(x_0).g'(x_0)$
- If we have: $f(x_0) \neq 0$, alors $\frac{1}{f}$ is differentiable at x_0 et $\left(\frac{1}{f}\right)'(x_0) = -\frac{f'(x_0)}{f(x_0)^2}$
- If we have: $g(x_0) \neq 0$, then $\frac{f}{g}$ is differentiable at x_0 and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0).g(x_0) - f(x_0).g'(x_0)}{g(x_0)^2}$$

Proposition : (On an interval)

Let f and g be two functions differentiable on an open interval I then:

- $f+g$ is differentiable on I and $(f+g)' = f' + g'$
- $f.g$ is differentiable on I and $(f.g)' = f'.g + f.g'$
- If $f \neq 0$ on I, $\dfrac{1}{f}$ is differentiable on I and $\left(\dfrac{1}{f}\right)' = -\dfrac{f'}{f^2}$
- If $g \neq 0$ on I, $\dfrac{f}{g}$ is differentiable on I and

$$\left(\frac{f}{g}\right)' = \frac{f'.g - f.g'}{g^2}$$

Proposition : Differentiability and composition

Let $f : I \longrightarrow \mathbb{R}$ and $g : J \longrightarrow \mathbb{R}$ be two functions where I and J are two open intervals such that: $f(I) \subset J$

- **Differentiability at a point:** If f is differentiable at x_0 and g is differentiable at $f(x_0)$, then $g \circ f$ is differentiable at x_0 and $(g \circ f)'(x_0) = f'(x_0).g'(f(x_0))$
- **differentiability on an interval:** If f is differentiable on I and g is differentiable on J, then $g \circ f$ is differentiable on I and $(g \circ f)' = f'.(g' \circ f)$

Proposition : Differentiability and inverse function

Let $f : I \longrightarrow J$ be a bijective and differentiable function at $x_0 \in I$. Then f^{-1} is differentiable at $y_0 = f(x_0)$ if and only if $f'(x_0) \neq 0$ and in this case: $(f^{-1})'(y_0) = \dfrac{1}{f'(x_0)}$.

Proposition

Let $f : I \longrightarrow J$ be a bijective and differentiable function on I. If $f' \neq 0$ on I, then f^{-1} is differentiable on J and we have : $(f^{-1})' = \dfrac{1}{f' \circ f^{-1}}$

Mean value Theorem

Theorem : (Rolle's theorem)

Let f be a function defined on $[a, b]$. If we have:

1. f is continuous on $[a, b]$.
2. f is differentiable on $]a, b[$
3. $f(a) = f(b)$

then there exists a real number $c \in]a, b[$ such that $f'(c) = 0$

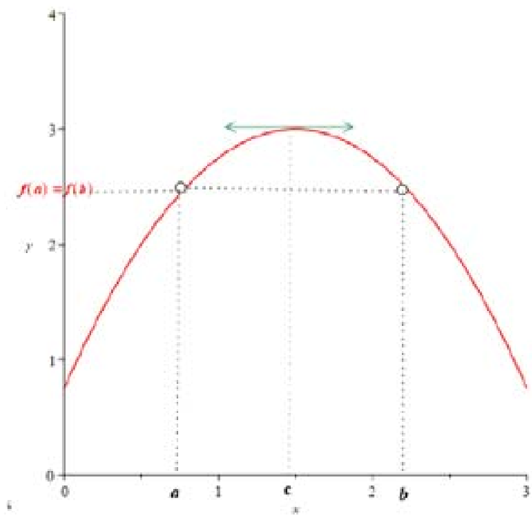

Figure : Geometrical interpretation of Rolle's theorem

Theorem : (Mean value Theorem)

Let f be a function defined on $[a, b]$, if we have:

1. f is continuous on $[a, b]$.
2. f is differentiable on $]a, b[$

then there exists a real number $c \in]a, b[$ such that:

$$f(b) - f(a) = f'(c)(b - a)$$

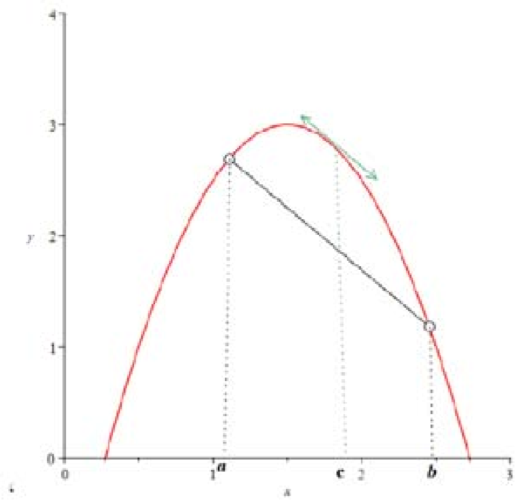

Figure : Geometrical interpretation of the mean value theorem

Consequence:(second form of the mean value theorem)

Let f be a function defined on I, $h > 0$ and $x_0 \in I$ such that $x_0 + h \in I$, then if we have:

1. f is continuous on $[x_0, x_0 + h]$.
2. f is derivable on $]x_0, x_0 + h[$

then there exists a $\theta \in]0, 1[$ such that:

$$f(x_0 + h) - f(x_0) = f'(x_0 + \theta.h)h$$

Example

By using the mean value theorem, show that:

$$\forall x > 0;\ \sin(x) \leq x$$

By putting $f(t) = t - \sin(t)$ we get:

$$\forall x > 0 \text{ we have:} \begin{cases} f \text{ is continuous on } [0, x] \\ \quad\quad \text{and} \\ f \text{ is differentiable on }]0, x[\end{cases}$$

According to the mean value theorem, there exists $c \in]0, x[$ such that:

$$f(x) - f(0) = f'(c)(x - 0)$$

$$\Longleftrightarrow x - \sin(x) = (1 - \cos(c))x \Longleftrightarrow \sin(x) = \cos(c)x$$

$$\Longrightarrow \sin(x) \leq x \text{ (as } \cos(c) \leq 1)$$

Theorem : Generalized mean value theorem

Let f and g be two real functions defined on $[a,b]$ such that:

1. f and g are continuous on $[a,b]$.
2. f and g are differentiable on $]a,b[$.

Then there exists a real number $c \in]a,b[$ such that:

$$(f(b)-f(a))g'(c) = (g(b)-g(a))f'(c)$$

Proposition (Variations of a function)

Let f be a continuous function on $[a,b]$ and differentiable on $]a,b[$, we have:

1. If $f'(x) > 0$ on $]a,b[$, then f is strictly increasing on $[a,b]$.
2. If $f'(x) \geq 0$ on $]a,b[$, then f is increasing on $[a,b]$.
3. If $f'(x) < 0$ on $]a,b[$, then f is strictly decreasing on $[a,b]$.
4. If $f'(x) \leq 0$ on $]a,b[$, then f is decreasing on $[a,b]$.
5. If $f'(x) = 0$ on $]a,b[$, then f is constant on $[a,b]$.

L'Hôpital's rule

Theorem

Let f and g be two continuous functions on I (where I is a neighborhood of x_0), differentiable on $I - \{x_0\}$ and satisfying the following conditions:

1. $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = 0$
2. $\forall x \in I - \{x_0\};\ g'(x) \neq 0$

Then:

$$\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = l \implies \lim_{x\to x_0} \frac{f(x)}{g(x)} = l$$

Example

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\cos(x)}{1} = 1$$

Remark *The converse is generally false. For example: $f(x) = x^2\cos(\frac{1}{x})$, $g(x) = x$. We have: $\lim\limits_{x\to 0} \frac{f(x)}{g(x)} = \lim\limits_{x\to 0} x\cos(\frac{1}{x}) = 0$. While $\lim\limits_{x\to 0} \frac{f'(x)}{g'(x)} = \lim\limits_{x\to 0}(2x\cos(\frac{1}{x}) + \sin(\frac{1}{x}))$ does not exist (since: $\lim\limits_{x\to 0} \sin(\frac{1}{x})$ does not exist)*

Remark *Also, the Hopital's rules is true when $x \to \pm\infty$*

2.2 Rules of Differentiation

Rule	**Formula**
Constant	$(c)' = 0$
Power	$(x^n)' = nx^{n-1}$
Sum	$(f+g)' = f' + g'$
Product	$(fg)' = f'g + fg'$
Quotient	$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
Chain Rule	$(f(g(x)))' = f'(g(x)) \cdot g'(x)$

Common Derivatives:

Function	Derivative
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
e^x	e^x
$\ln x$	$\frac{1}{x}$

Chapter 3

Integrals

Integration is one of the two main operations in calculus (the other is differentiation). While differentiation finds the *rate of change*, integration finds the *total quantity* or the *area under a curve*.

Integration is used in biology to calculate:

- Total bacterial growth over time,
- Total oxygen consumption,
- Accumulated concentrations of a substance.

If we know a function $f(x)$ that represents a rate (for example, growth rate), the **integral** of $f(x)$ between a and b gives the total change:

$$\text{Area under } y = f(x) \text{ between } x = a \text{ and } x = b.$$

3.1 Indefinite Integral

The **indefinite integral** of $f(x)$ is a function $F(x)$ such that:

$$F'(x) = f(x)$$

and we write:

$$\int f(x)\,dx = F(x) + C$$

where C is the constant of integration.

Examples

In the following c denote a real constant ($c \in \mathbb{R}$).

1. $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$, with $n \neq -1$.
2. $\int x^{-1} dx = \int \frac{1}{x} dx = \ln(|x|) + c$.
3. $\int e^x dx = e^x + c$.
4. $\int a^x dx = \frac{a^x}{\ln a} + c$.

5. $\int \sin(x)dx = -\cos(x) + c.$
6. $\int \cos(x)dx = \sin(x) + c.$
7. $\int \frac{1}{\sin^2(x)}dx = -ctan(x) + c.$
8. $\int \frac{1}{\cos^2(x)}dx = \tan(x) + c.$
9. $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x) + c.$
10. $\int \frac{1}{1+x^2}dx = \arctan(x) + c.$
11. $\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\left(\frac{x}{a}\right) + c.$
12. $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan(\frac{x}{a}) + c.$
13. $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\left(\left|\frac{a-x}{a+x}\right|\right) + c.$
14. $\int \frac{1}{\sqrt{x^2 \pm a^2}}dx = \ln\left(\left|x + \sqrt{a^2 \pm x^2}\right|\right) + c$

3.2 Definite Integral

The **definite integral** gives a numerical value representing the area under the curve between a and b:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Example

$$\int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

3.3 Properties of Integrals

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

3.4 Integration Techniques

Substitution Method

If $u = g(x)$, then:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Example:

$$\int 2xe^{x^2}\,dx$$

Let $u = x^2 \Rightarrow du = 2x\,dx$

$$\int e^u du = e^u + C = e^{x^2} + C$$

$\int \frac{e^{-x}}{\sqrt{1-e^{-2x}}}dx = ??$
To compute this integral we use the following substitution
we put

$$t = e^{-x} \Longrightarrow dt = -e^{-x}dx$$

so

$$\begin{aligned} \int \frac{e^{-x}}{\sqrt{1-e^{-2x}}}dx &= \int \frac{-1}{\sqrt{1-t^2}}dt = \arccos(t) + c = \\ &= \arccos\left(e^{-x}\right) + c = -\arcsin\left(e^{-x}\right) + c \qquad \text{with } c \in \mathbb{R}. \end{aligned}$$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Example:

$$\int xe^x dx$$

Let $u = x \Rightarrow du = dx$, and $dv = e^x dx \Rightarrow v = e^x$

$$\int xe^x dx = xe^x - \int e^x dx = e^x(x-1) + C$$

To compute the integral we use the integration by parts method. For this, we take the following:

$$\begin{cases} u &= x+2 \\ v' &= \cos(2x) \end{cases} \Longrightarrow \begin{cases} u' &= 1 \\ v &= \frac{1}{2}\sin(2x) \end{cases}$$

Thus,

$$\begin{aligned} \int (x+2)\cos(2x)dx &= \frac{1}{2}(x+2)\sin(2x) - \int \frac{1}{2}\sin(2x)dx \\ &= \frac{1}{2}(x+2)\sin(2x) + \frac{1}{4}\cos(2x) + c \qquad \text{with } c \in \mathbb{R}. \end{aligned}$$

Integration of Rational Functions

In this section we will take a more detailed look at the use of partial fraction decompositions in evaluating integrals of rational functions, a technique we first encountered in the inhibited growth model example in the previous section.
We begin with a few examples to illustrate how some integration problems involving rational functions may be simplified either by a long division or by a simple substitution.

Example To evaluate $\int \frac{x^2}{x+1}\,dx$, we first perform a long division of $x+1$ into x^2 to obtain

$$\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}.$$

Then

$$\int \frac{x^2}{x+1}\,dx = \int \left(x - 1 + \frac{1}{x+1}\right) dx = \frac{1}{2}x^2 - x + \log|x+1| + c.$$

Example To evaluate $\int \frac{2x+1}{x^2+x}\,dx$, we make the substitution

$$\begin{aligned} u &= x^2 + x \\ du &= (2x+1)dx. \end{aligned}$$

Then

$$\int \frac{2x+1}{x^2+x}\,dx = \int \frac{1}{u}\,du = \log|u| + c = \log|x^2+x| + c.$$

Example To evaluate $\int \frac{x}{x+1}\,dx$, we perform a long division of $x+1$ into x to obtain

$$\frac{x}{x+1} = 1 - \frac{1}{x+1}.$$

Then

$$\int \frac{x}{x+1}\,dx = \int \left(1 - \frac{1}{x+1}\right) dx = x - \log|x+1| + c.$$

Alternatively, we could evaluate this integral with the substitution

$$\begin{aligned} u &= x + 1 \\ du &= dx. \end{aligned}$$

With this substitution, $x = u - 1$, so we have

$$\begin{aligned}\int \frac{x}{x+1}\,dx &= \int \frac{u-1}{u}\,du \\ &= \int \left(1 - \frac{1}{u}\right)\,du \\ &= u - \log|u| + c \\ &= x + 1 - \log|x+1| + c.\end{aligned}$$

Note that this is the same answer we obtained above, although with a different constant of integration.

Partial fraction decomposition: Distinct linear factors

Now we consider the general problem of evaluating

$$\int \frac{f(x)}{g(x)}\,dx$$

where both f and g are polynomials. We will assume that the degree of g is less than the degree of f. As illustrated in the first and third examples above, if this is not the case, we can first perform a long division to simplify the quotient into the form of a polynomial plus a remainder term which is a rational function with numerator of degree less than the denominator. To begin we will suppose that g factors completely into n distinct linear factors. That is, suppose there are constants $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ such that

$$g(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_nx + b_n), \tag{6.4.1}$$

where the factors on the right are all distinct. From a theorem of linear algebra, which we will not attempt to prove here, there exist constants $A_1, A_2, \ldots, A_n$ such that

$$\frac{f(x)}{g(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}. \tag{6.4.2}$$

The expression on the right of (6.4.2) is called the *partial fraction decomposition* of $\dfrac{f(x)}{g(x)}$. Once the constants $A_1, A_2, \ldots, A_n$ are determined, the evaluation of

$$\int \frac{f(x)}{g(x)}\,dx$$

becomes a routine problem. The next examples will illustrate one method for finding these constants.

Example To evaluate $\displaystyle\int \frac{1}{(x-2)(x-3)}\,dx$, we need to find constants A and B such that

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}.$$

Combining the terms on the right, we have

$$\frac{1}{(x-2)(x-3)} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}.$$

Now two rational functions with equal denominators are equal only if their numerators are also equal; hence we must have

$$1 = A(x-3) + B(x-2)$$

for all values of x. In particular, for $x = 2$ we obtain

$$1 = -A,$$

from which it follows that $A = -1$, and for $x = 3$ we have

$$1 = B.$$

Thus

$$\frac{1}{(x-2)(x-3)} = -\frac{1}{x-2} + \frac{1}{x-3},$$

so

$$\begin{aligned}\int \frac{1}{(x-2)(x-3)}\,dx &= -\int \frac{1}{x-2}\,dx + \int \frac{1}{x-3}\,dx \\ &= -\log|x-2| + \log|x-3| + c.\end{aligned}$$

Example To evaluate $\displaystyle\int \frac{3x}{(x+5)(2x-1)}\,dx$, we need to find constants A and B such that

$$\frac{3x}{(x+5)(2x-1)} = \frac{A}{x+5} + \frac{B}{2x-1}.$$

Combining the terms on the right, we have

$$\frac{3x}{(x+5)(2x-1)} = \frac{A(2x-1)+B(x+5)}{(x+5)(2x-1)}.$$

As before, it follows that

$$3x = A(2x-1) + B(x+5)$$

for all values of x. In particular, for $x = -5$ we obtain

$$-15 = -11A,$$

from which it follows that

$$A = \frac{15}{11},$$

and for $x = \frac{1}{2}$ we have

$$\frac{3}{2} = \frac{11}{2}B,$$

from which it follows that

$$B = \frac{3}{11}.$$

Hence

$$\frac{3x}{(x+5)(2x-1)} = \frac{15}{11}\frac{1}{x+5} + \frac{3}{11}\frac{1}{2x-1},$$

so

$$\begin{aligned}\int \frac{1}{(x+5)(2x-1)}\,dx &= \frac{15}{11}\int \frac{1}{x+5}\,dx + \frac{3}{11}\frac{1}{2x-1}\,dx\\ &= \frac{15}{11}\log|x+5| + \frac{3}{22}\log|2x-1| + c.\end{aligned}$$

Partial fraction decomposition: Repeated linear factors

Returning to the general problem of evaluating

$$\int \frac{f(x)}{g(x)}\,dx,$$

where f and g are both polynomials and the degree of f is less than the degree of g, we will now consider the case where g factors completely into linear factors, allowing for the possibility that one or more of these factors may be repeated. Specifically, suppose the factor $ax + b$ occurs n times in the factorization of g. Then the partial fraction decomposition of $\frac{f(x)}{g(x)}$ must contain a sum of terms of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}, \tag{6.4.3}$$

for some constants $A_1, A_2, \ldots, A_n$, in addition to similar terms for every other factor of g. This is best illustrated in an example.

Example To evaluate $\frac{x+1}{(x-1)^3(x-2)}\,dx$, we need to find constants A, B, C, and D such that

$$\frac{x+1}{(x-1)^3(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x-2}. \tag{6.4.4}$$

That is, this partial fraction decomposition contains three terms corresponding to the factor $x-1$, since it is repeated three times, and only one term corresponding to the factor $x-2$, since it occurs only once. Moreover, the degrees of the denominators of the terms for $x-1$ increase from 1 to 3. Now combining the terms on the right of (6.4.4), we have

$$\frac{x+1}{(x-1)^3(x-2)} = \frac{A(x-1)^2(x-2) + B(x-1)(x-2) + C(x-2) + D(x-1)^3}{(x-1)^3(x-2)}.$$

Again, it follows that

$$x + 1 = A(x-1)^2(x-2) + B(x-1)(x-2) + C(x-2) + D(x-1)^3 \tag{6.4.5}$$

for all values of x. However, because of the repeated factors, we cannot choose values for x which will isolate each of the constants one at a time as we did in the previous examples. Instead, we will illustrate another technique for finding the constants. By multiplying out (6.4.5) and collecting terms, we obtain

$$\begin{aligned} x+1 &= A(x^3 - 4x^2 + 5x - 2) + B(x^2 - 3x + 2) + C(x-2) + D(x^3 - 3x^2 + 3x - 1) \\ &= (A+D)x^3 + (-4A + B - 3D)x^2 + (5A - 3B + C + 3D)x - 2A + 2B - 2C - D \end{aligned}$$

for all values of x. Since two polynomials are equal only if they have equal coefficients, we can equate the coefficients of $x + 1$ with the coefficients of the polynomial on the right to obtain the four equations

$$\begin{aligned} A + D &= 0 \\ -4A + B - 3D &= 0 \\ 5A - 3B + C + 3D &= 1 \\ -2A + 2B - 2C - D &= 1. \end{aligned} \tag{6.4.6}$$

From the first equation we learn that

$$D = -A.$$

Substituting this into the second equation gives us

$$B = A.$$

Substituting both of these values into the third equation results in

$$C = A + 1.$$

Finally, substituting for D, B, and C in the fourth equation gives us

$$-2A + 2A - 2(A+1) + A = 1,$$

which gives us $A = -3$. Hence $B = -3$, $C = -2$, and $D = 3$. Thus

$$\begin{aligned} \int \frac{x+1}{(x-1)^3(x-2)}\,dx &= -\int \frac{3}{(x-1)}\,dx - \int \frac{3}{(x-1)^2}\,dx \\ &\quad - \int \frac{2}{(x-1)^3}\,dx + \int \frac{3}{x-2}\,dx \\ &= -3\log|x-1| + \frac{3}{x-1} + \frac{1}{(x-1)^2} + 3\log|x-2| + c. \end{aligned}$$

Example: To evaluate the following integral:

$$\int \frac{1}{(x^2+1)(x-1)}\,dx$$

we need to find constants A, B and C such that:

$$\frac{1}{(x^2+1)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

By multiplying both sides by$(x^2+1)(x-1)$, we obtain :

$$1 = A(x^2+1) + (Bx+C)(x-1).$$

Let's develop:

$$1 = A(x^2+1) + Bx(x-1) + C(x-1) = Ax^2 + A + Bx^2 - Bx + Cx - C.$$

Let's group similar terms together:

$$1 = (A+B)x^2 + (-B+C)x + (A-C).$$

By identifying the coefficients :

$$\begin{cases} A+B=0, \\ -B+C=0, \\ A-C=1. \end{cases}$$

We solve the system :

$$C = B, \quad A = -B, \quad \Rightarrow A - C = -B - B = -2B = 1 \Rightarrow B = -\frac{1}{2}.$$

So :

$$A = \frac{1}{2}, \quad B = -\frac{1}{2}, \quad C = -\frac{1}{2}.$$

The decomposition into partial fractions is therefore:

$$\frac{1}{(x^2+1)(x-1)} = \frac{1/2}{x-1} - \frac{1}{2}\cdot\frac{x+1}{x^2+1}.$$

So:

$$\int \frac{1}{(x^2+1)(x-1)}\,dx = \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{x+1}{x^2+1}\,dx.$$

We separate the second integral:

$$\int \frac{x+1}{x^2+1}\,dx = \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) + \tan^{-1}(x).$$

Substitution in the integral :

$$\int \frac{1}{(x^2+1)(x-1)}\,dx = \frac{1}{2}\ln|x-1| - \frac{1}{2}\left(\frac{1}{2}\ln(x^2+1) + \arctan(x)\right) + C.$$

Let's simplify :

$$\boxed{\int \frac{1}{(x^2+1)(x-1)}\,dx = \frac{1}{2}\ln|x-1| - \frac{1}{4}\ln(x^2+1) - \frac{1}{2}\arctan(x) + C.}$$

Example: To evaluate the following integral:

$$\int \frac{x^3 + 2x^2 + 3}{x^2 + 1}\, dx$$

Perform long division:

$$\frac{x^3 + 2x^2 + 3}{x^2 + 1} = x + 2 + \frac{-x + 1}{x^2 + 1}.$$

Then integrate term by term:

$$\int \left(x + 2 + \frac{-x + 1}{x^2 + 1} \right) dx = \frac{x^2}{2} + 2x + \int \frac{-x}{x^2 + 1}\, dx + \int \frac{1}{x^2 + 1}\, dx.$$

$$\int \frac{-x}{x^2 + 1}\, dx = -\frac{1}{2} \ln(x^2 + 1), \qquad \int \frac{1}{x^2 + 1}\, dx = \arctan(x).$$

$$\boxed{\int \frac{x^3 + 2x^2 + 3}{x^2 + 1}\, dx = \frac{x^2}{2} + 2x - \frac{1}{2} \ln(x^2 + 1) + \arctan(x) + C.}$$