

Chapter 2. Study of circuits in transient regime

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Chapter 2. Study of circuits in transient regime

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Introduction

In the study of electrical circuits, we have often considered permanent conditions, i.e. when electrical quantities (voltage, current) remain constant over time in continuous regime, or vary regularly in sinusoidal regime.

However, when a circuit is subjected to a sudden change (e.g., closing or opening a switch, charging or discharging a capacitor, energising a coil, etc.), it goes through a special phase called transient regime.

During this regime, voltages and currents change gradually before reaching their new steady state. Studying this behaviour is essential because it allows us to:

- Understand the role of inductance and capacitance in circuit dynamics,
- Analyse the time response of electrical systems,
- Predict important phenomena such as overvoltages, response times and circuit stability.

II.1 Equation of transient regimes

Since electrical quantities are variable and their form is not known a priori, it is necessary to use the operating equations for elementary dipoles. Figure II.1 shows these equations for the three most commonly used linear passive dipoles.

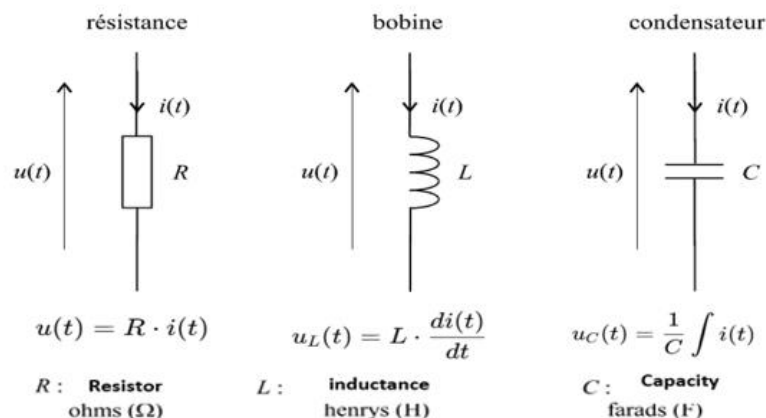


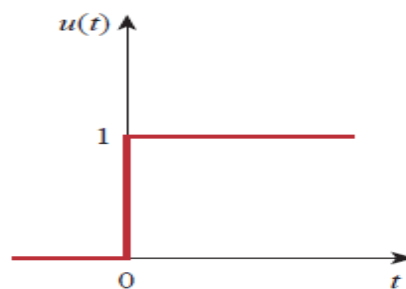
Figure II.1

In a linear circuit under any operating condition, Kirchhoff's laws and Millman's theorem are the most commonly used tools. Since the concept of impedance is reserved for sinusoidal conditions, we cannot use it here. It will therefore be difficult to use Thévenin's and Norton's theorems in transient conditions.

Writing Kirchhoff's laws in a transient circuit generates more complex equations than in continuous or sinusoidal conditions. These are generally linear differential equations with constant coefficients.

Here, we will limit ourselves to studying first- and second-order differential equations, which are most commonly encountered in problems related to transient regimes in linear electrical circuits. For more complex equations, it would be necessary to use more sophisticated mathematical tools, which are beyond the scope of this document.

In this chapter, we focus on linear circuits, particularly RC, RL, series circuits. These circuits are subjected to a step signal (a signal whose value changes from one level, usually 0, to another level E in zero time)



II.2 RC circuit free regime

II.2.1 Voltage variation across the capacitor terminals

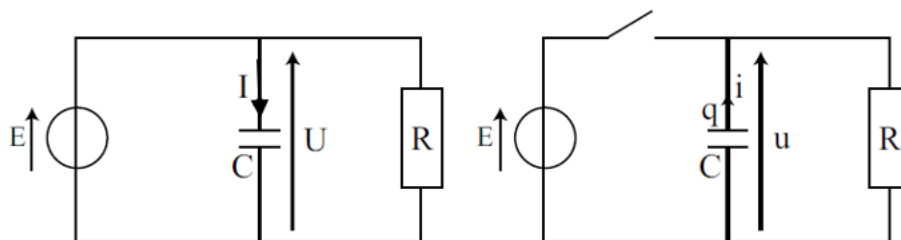


Figure II .2 A source-free RC circuit

A source-free RC circuit occurs when its DC source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors. Consider a series combination of a resistor and an initially charged capacitor, as shown in Figure II.1 (The resistor and capacitor may be the equivalent resistance and equivalent capacitance of combinations of resistors and capacitors.) Our objective is to determine the circuit response, which, for pedagogic reasons,

we assume to be the voltage $v(t)$ across the capacitor. Since the capacitor is initially charged, we can assume that at time $t = 0$ the initial voltage is:

$$v(0) = V_0 \quad (\text{II.1})$$

with the corresponding value of the energy stored as:

$$w(0) = \frac{1}{2} C V_0^2 \quad (\text{II.2})$$

Application of Kirchhoff's law to the superior nodes of the circuit in

$$i_C + i_R = 0 \quad (\text{II.3})$$

By definition, $i_C = C \frac{dv}{dt}$ and $i_R = \frac{v}{R}$. Thus

$$C \frac{dv}{dt} + \frac{v}{R} = 0 \quad (\text{II.4a})$$

Or:

$$\frac{dv}{dt} + \frac{v}{RC} = 0 \quad (\text{II.4b})$$

This is a *first-order differential equation*, since only the first derivative of v is involved. To solve it, we rearrange the terms as:

$$\frac{dv}{v} = -\frac{1}{RC} dt \quad (\text{II.5})$$

Integrating both sides, we get

$$\ln v = -\frac{t}{RC} + \ln A$$

Where is the integration constant. Thus,

$$\ln \frac{v}{A} = -\frac{t}{RC} \quad (\text{II.6})$$

Taking powers of e produces

$$v(t) = A e^{-t/RC} \quad (\text{II.7})$$

With: $\tau = RC$

But from the initial conditions, $v(0) = A = V_0$ Hence,

$$v(t) = V_0 e^{-t/\tau} \quad (\text{II.8})$$

This shows that the voltage response of the RC circuit is an exponential decay of the initial voltage. Since the response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called the *natural response* of the circuit.

The natural response is illustrated graphically in Fig. II.3. Note that at $t=0$ we have the correct initial condition as in Eq. (II.1). As t increases, the voltage decreases toward zero. The rapidity with which the voltage decreases is expressed in terms of the *time constant*, denoted by τ , the lowercase Greek letter tau.

The **time constant** of a circuit is the time required for the response to decay to a factor of $1/e$ or 36.8 percent of its initial value.¹

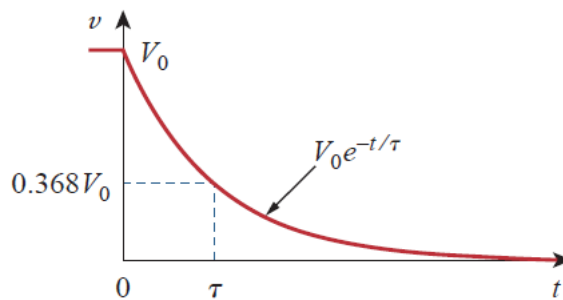


Figure II. 3
The voltage response of the RC circuit

This implies that at $t = \tau$; Eq. (II.8) becomes:

$$V_0 e^{-\tau/RC} = V_0 e^{-1} = 0.368V_0$$

Values of $v(t)/V_0 = e^{-t/\tau}$	
t	$v(t)/V_0$
τ	0.36788
2τ	0.13534
3τ	0.04979
4τ	0.01832
5τ	0.00674

Table II.1

With a calculator it is easy to show that the value of $v(t)/V_0$ is as shown in Table II.1. It is evident from Table II.1 that the voltage $v(t)$ is less than 1% of after 5τ . Thus, it is customary to assume that the capacitor is fully discharged (or charged) after five time constants. In other words, it takes 5τ for the circuit to reach its final state or steady state when no changes take place with time. Notice that for every time interval of τ the voltage is reduced by 36.8% of its previous value, $v(t + \tau) = \frac{v(t)}{e} = 0.368 v(t)$, regardless of the value of t .

Observe τ that the smaller the time constant, the more rapidly the voltage decreases, that is, the faster the response. This is illustrated in Fig.II.4. A circuit with a small time constant gives a fast response in that it reaches the steady state (or final state) quickly due to quick dissipation of energy stored, whereas a circuit with a large time constant gives a slow response because it

takes longer to reach steady state. At any rate, whether the time constant is small or large, the circuit reaches steady state in five time constants.

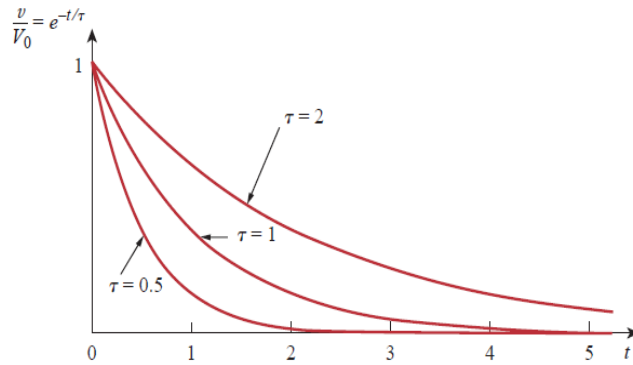


Figure.II.4. Plot of $v/V_0 = e^{-t/\tau}$ for various values of the time constant

II.2.2 Evolution of current intensity

With the voltage $v(t)$ in Eq. (II.8), we can find the current $i_R(t)$:

$$i_R = \frac{v(t)}{R} = \frac{V_0}{R} e^{-t/\tau} \quad (\text{II.9})$$

II.2.3 Energy study

The power dissipated in the resistor is:

$$p(t) = v i_R = \frac{V_0^2}{R} e^{-2t/\tau} \quad (\text{II.10})$$

The energy absorbed by the resistor up to time t is:

$$\begin{aligned} w_R(t) &= \int_0^t p dt = \int_0^t \frac{V_0^2}{R} e^{-2t/\tau} dt \\ &= -\frac{\tau V_0^2}{2R} e^{-2t/\tau} \Bigg|_0^t = \frac{1}{2} C V_0^2 (1 - e^{-2t/\tau}), \tau = RC \end{aligned} \quad (\text{II.11})$$

II.3 The Source-Free RL Circuit

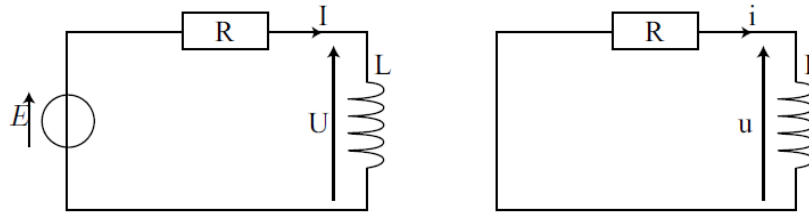


Figure II.5. A source-free RL circuit.

II.3.1 Evolution of current intensity

Consider the series connection of a resistor and an inductor, as shown in Fig. II.5. Our objective is to determine the circuit response, which we will assume to be the current through the inductor. We select the inductor current as the response in order to take advantage of the idea that the inductor current cannot change instantaneously. At $t=0$ we assume that the inductor has an initial current I_0 or

$$i(0) = I_0 \quad (\text{II.12})$$

With the corresponding energy stored in the inductor as

$$w(0) = \frac{1}{2} L I_0^2 \quad (\text{II.13})$$

Applying Kirchhoff's Voltage Law around the loop in Fig.II.5

$$v_L + v_R = 0 \quad (\text{II.14})$$

But: $v_L = L \frac{di}{dt}$, and $v_R = iR$. Thus:

$$L \frac{di}{dt} + iR = 0$$

Or

$$\frac{di}{dt} + \frac{R}{L} i = 0 \quad (\text{II.15})$$

Rearranging terms and integrating gives

$$\int_{I_0}^{i(t)} \frac{di}{i} = - \int_0^t \frac{R}{L} dt \quad (\text{II.16})$$

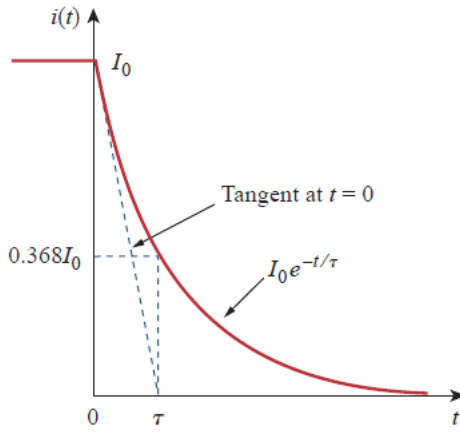
$$\ln i \Big|_{I_0}^{i(t)} = - \frac{Rt}{L} \Big|_0^t \Rightarrow \ln i(t) - \ln I_0 = - \frac{Rt}{L} + 0$$

$$\ln \frac{i(t)}{I_0} = -\frac{Rt}{L} \quad (\text{II.17})$$

Taking the powers of e , we have

$$i(t) = I_0 e^{-Rt/L} \quad (\text{II.18})$$

This shows that the natural response of the RL circuit is an exponential decay of the initial current. The current response is shown in Fig.II.6. It is evident from Eq. (II.18) that the time constant for the RL circuit is



$$\tau = \frac{L}{R} \quad (\text{II.19})$$

Figure II.6. The current response of the RL circuit

With τ again having the unit of seconds. Thus, Eq. (II.18) may be written as

$$i(t) = I_0 e^{-t/\tau} \quad (\text{II.20})$$

II.3.2 Voltage variation across the coil terminals

With the current in Eq. (II.20), we can find the voltage across the resistor as

$$v_R(t) = iR = I_0 R e^{-t/\tau} \quad (\text{II.21})$$

II.3.3 Energy study

The power dissipated in the resistor is

$$p = v_R i = I_0^2 R e^{-2t/\tau} \quad (\text{II.22})$$

The energy absorbed by the resistor is

$$w_R(t) = \int_0^t p dt = \int_0^t I_0^2 R e^{-2t/\tau} dt = -\frac{1}{2} \tau I_0^2 R e^{-2t/\tau} \Big|_0^t$$

Or

$$w_R(t) = -\frac{1}{2}LI_0^2 \left(1 - e^{-2t/\tau}\right) \quad (\text{II.23})$$

II.4 Response of an RC circuit to a voltage step

When the DC source of an RC circuit is suddenly applied, the voltage or current source can be modeled as a step function, and the response is known as a *step response*.

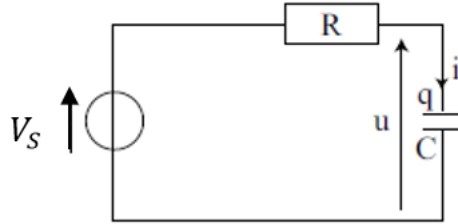


Figure II.7. An RC circuit with voltage step input.

II.4.1 Voltage variation across the capacitor terminals

The step response is the response of the circuit due to a sudden application of a dc voltage or current source.

Consider the RC circuit in Fig.II.7. Where is a constant dc voltage source, we select the capacitor voltage as the circuit response to be determined, we assume an initial voltage on the capacitor, although this is not necessary for the step response. Since the voltage of a capacitor cannot change instantaneously.

$$V_S = Ri + u = RC \frac{du}{dt} + u$$

$$\frac{du}{dt} + \frac{u}{RC} = \frac{V_S}{RC} \quad (\text{II.24})$$

Rearranging terms gives

$$\frac{du}{dt} = -\frac{u - V_S}{RC}$$

Or

$$\frac{du}{u - V_S} = -\frac{dt}{RC} \quad (\text{II.25})$$

Integrating both sides and introducing the initial conditions

$$\ln(u - V_S) \Big|_{V_0}^{v(t)} = -\frac{t}{RC} \Big|_0^t$$

$$\ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

$$\ln \frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC} \quad (\text{II.26})$$

Taking the exponential of both sides

$$v - V_s = (V_0 - V_s)e^{-t/\tau}$$

Or

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \quad t > 0 \quad (\text{II. 27})$$

$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases} \quad (\text{II.28})$$

This is known as the *complete response* (or total response) of the *RC* circuit to a sudden application of a dc voltage source, assuming the capacitor is initially charged. The reason for the term “complete” will become evident a little later. Assuming that $V_s > 0$, a plot of $v(t)$ is shown in Fig. II.8.

If we assume that the capacitor is uncharged initially, we set in Eq. (II.28) so that

$$v(t) = \begin{cases} 0 & t < 0 \\ V_s(1 - e^{-t/\tau}) & t > 0 \end{cases} \quad (\text{II. 29})$$

Which can be written alternatively as

$$v(t) = V_s(1 - e^{-t/\tau})$$

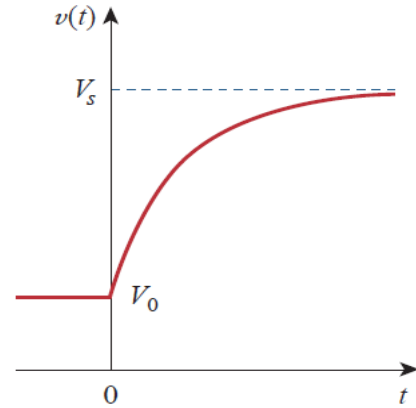


Figure II.8. Response of an *RC* circuit with initially charged capacitor.

This is the complete step response of the *RC* circuit when the capacitor is initially uncharged.

II.4.2 Evolution of current intensity

The current through the capacitor is obtained from Eq. (II.29) using $i_C = C \frac{dv}{dt}$. we get

$$i(t) = C \frac{dv}{dt} = \frac{C}{\tau} V_s e^{-t/\tau}, \quad t > 0$$

$$i(t) = \frac{V_s}{R} e^{-t/\tau} \quad (\text{II.30})$$

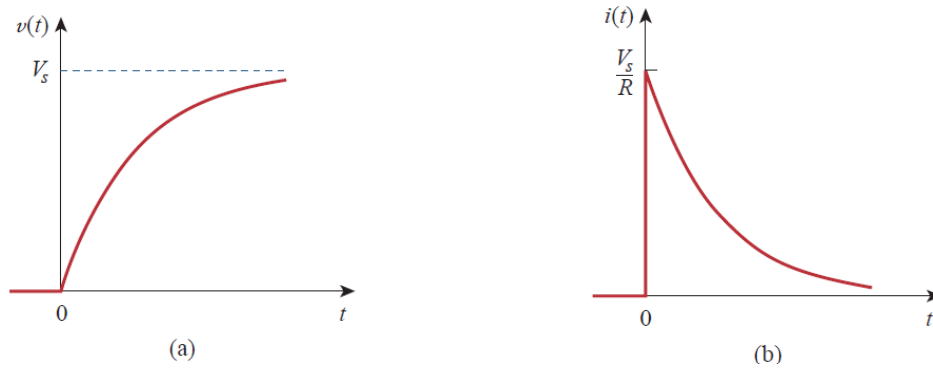


Figure II.9. Step response of an RC circuit with initially uncharged capacitor: (a) voltage response, (b) current response.

Rather than going through the derivations above, there is a systematic approach—or rather, a short-cut method—for finding the step response of an RC or RL circuit. Let us reexamine Eq. (II.27), which is more general than Eq. (II.29). It is evident that it has two components.

Classically there are two ways of decomposing this into two components.

The first is to break it into a “natural response and a forced response” and the second is to break it into a “transient response and a steady-state response.” Starting with the natural response and forced response, we write the total or complete response as:

$\text{Complete response} = \underset{\text{stored energy}}{\text{natural response}} + \underset{\text{independent source}}{\text{forced response}}$
--

Or

$$v = v_n + v_f \quad (\text{II.31})$$

Where

$$v_n = V_0 e^{-t/\tau}$$

And

$$v_f = V_s (1 - e^{-t/\tau})$$

II.4.3 Energy study

Multiply $V_s = Ri + u$ by i give:

$$V_s i = Ri^2 + ui$$

Where $V_s i$ is the power supplied by the generator ($V_s (-i)$ power received); Ri^2 is the power received and dissipated in the resistor; ui is the power received and stored in the capacitor. The energy supplied by the generator is distributed equally between the resistor and the capacitor.

$$\int_0^\infty V_s i dt = \frac{V_s^2}{R} \int_0^\infty e^{(-t/\tau)} dt = \frac{V_s^2}{R} RC = CV_s^2$$

$$\int_0^\infty Ri^2 dt = R \frac{V_s^2}{R^2} \int_0^\infty e^{(-2t/\tau)} dt = R \frac{V_s^2}{R^2} \frac{RC}{2} = \frac{1}{2} CV_s^2$$

$$\begin{aligned} \int_0^\infty u i dt &= \frac{V_s^2}{R^2} \int_0^\infty (e^{(-t/\tau)} - e^{(-2t/\tau)}) dt = \frac{V_s^2}{R} \left(RC - \frac{RC}{2} \right) \\ &= \frac{1}{2} CV_s^2 \end{aligned}$$

II.5 Response of an RL circuit to a voltage step

Consider the RL circuit in Fig.II.10, our goal is to find the inductor current I as the circuit response.

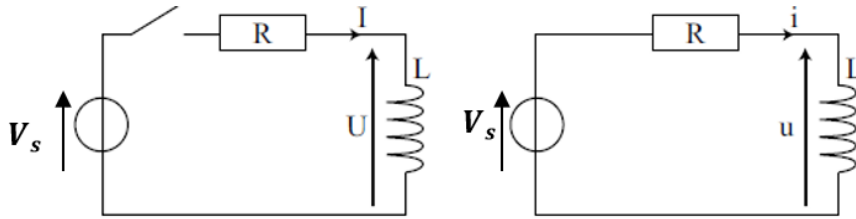


Figure II.10. An RL circuit with a step input voltage.

II.5.1 Current intensity evolution

Rather than apply Kirchhoff's laws, Let the response be the sum of the transient response and the steady-state response

$$i = i_t + i_{ss} \quad (\text{II.32})$$

We know that the transient response is always a decaying exponential, that is,

$$i_t = Ae^{-t/\tau}, \tau = L/R \quad (\text{II.33})$$

The steady-state response is the value of the current a long time after the switch in Fig. II.10 is closed. We know that the transient response essentially dies out after five time constants. At

that time, the inductor becomes a short circuit, and the voltage across it is zero. The entire source voltage appears across R . Thus, the steady-state response is:

$$i_{ss} = \frac{V_s}{R} \quad (\text{II.34})$$

Substituting Eq. (II.33) and (II.34) into Eq. (II.32) gives

$$i = Ae^{-t/\tau} + \frac{V_s}{R} \quad (\text{II.35})$$

We now determine the constant A from the initial value of i . Let I_0 be the initial current through the inductor, which may come from a source other than V_s . Since the current through the inductor cannot change instantaneously,

Thus, at $t = 0$, Eq. (II.35) becomes:

$$I_0 = A + \frac{V_s}{R} \quad (\text{II.36})$$

From this, we obtain A as:

$$A = I_0 - \frac{V_s}{R}$$

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-t/\tau} \quad \text{This is the complete response of the RL circuit. It is illustrated in fig.II.11.} \quad (\text{II.37})$$

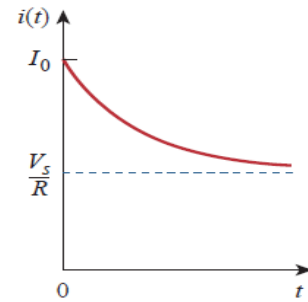


Figure II.11.Total response of the RL circuit with initial inductor current I_0 .

This is the complete response of the RL circuit. The response in Eq. (II.37) may be written as:

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

If $I_0 = 0$, then

$$i(t) = \begin{cases} 0 & t < 0 \\ \frac{V_s}{R}(1 - e^{-t/\tau}) & t > 0 \end{cases} \quad (\text{II.37a})$$

$$i(t) = \frac{V_s}{R}(1 - e^{-t/\tau}) \quad (\text{II.37b})$$

This is the step response of the RL circuit with no initial inductor current.

II.5.2 Voltage variation across the coil terminals

The voltage across the inductor is obtained from Eq. (II.37) using $v = L di/dt$ We get

$$v(t) = L \frac{di}{dt} = V_s \frac{L}{\tau R} e^{-t/\tau}, \quad \tau = \frac{L}{R}, \quad t > 0$$

Or

$$v(t) = V_s e^{-t/\tau} \quad (\text{II.38})$$

Figure II.12. Shows the step responses in Eqs. (II.37) and (II.38)

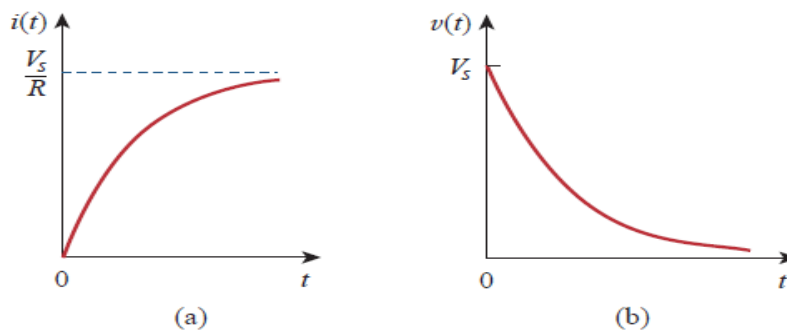


Figure II.12. Step responses of an RL circuit with no initial inductor current: (a) current response, (b) voltage response.

II.5.3 Energy study

$$V_s i = Ri^2 + ui$$

Where $V_s i$ is the power supplied by the generator ($V_s(-i)$ power received); Ri^2 is the power received and dissipated in the resistor; ui is the power received and stored in the coil. The energy supplied by the generator is distributed equally between the resistor and the capacitor.

When $t \rightarrow \infty$ A new steady state is established with $I = V_s/R$, therefore:

$$\begin{aligned} \int_0^\infty V_s i dt &\rightarrow \infty \\ \int_0^\infty Ri^2 dt &\rightarrow \infty \\ \int_0^\infty ui dt &= \frac{V_s^2}{R} \int_0^\infty (e^{-t/\tau} - e^{-2t/\tau}) dt = \frac{V_s^2}{R} \left(\frac{L}{R} - \frac{L}{2R} \right) = \frac{1}{2} LI^2 \end{aligned}$$

