Chapter 1

Numerical Methods for Finding Roots

This chapter provides detailed formulas and derivations (without full formal proofs) for four classical root-finding methods: Fixed Point Iteration, Bisection, Newton–Raphson and Secant methods. For each method we give 2 examples with Python implementations and numerical results (iterations, final approximations, absolute error, and observed order of convergence when possible).

1.1 Fixed Point Iteration Method

1.1.1 Method and Derivation (brief)

Given $f(x) = 0$ we rewrite as $x = g(x)$. The iteration is

$$x_{n+1} = g(x_n). \tag{1.1}$$

If g is continuously differentiable in a neighborhood of a fixed point x^* and $|g'(x^*)| < 1$, then the sequence converges linearly. A first-order error relation (Taylor expansion) gives:

$$e_{n+1} := x_{n+1} - x^* = g(x_n) - g(x^*) = g'(x^*)(x_n - x^*) + \mathcal{O}((x_n - x^*)^2), \tag{1.2}$$

hence asymptotically $|e_{n+1}| \approx |g'(x^*)|\,|e_n|$.

1.2 Example 1: $f(x) = \cos x - x$, $g(x) = \cos x$

The root is $x^* \approx 0.7390851332151607$.

Iteration and results

Initial guess	$x_0 = 0.5$
Iterations performed	69
Final approximation	$x_n = 0.739085133215$
Absolute error	$\lvert x_n - x^*\rvert = 3.063e - 13$
Observed order (estimate)	1.000

1.3 Example 2: $f(x) = x^3 - x - 2$, transformation $g(x) = (x+2)^{1/3}$

This transformation is not standard; convergence is not guaranteed for all initial guesses but can work near the root $x^* \approx 1.5213797068045676$.

Iteration and results

Initial guess	$x_0 = 1.5$
Iterations performed	14
Final approximation	$x_n = 1.521379706805$
Absolute error	$\lvert x_n - x^* \rvert = 3.531e - 14$
Observed order (estimate)	1.000

1.4 Bisection Method

1.4.1 Method and Error Estimate (brief)

For continuous f with $f(a)f(b) < 0$ the intermediate value theorem guarantees a root in (a, b). Bisection halves the interval each iteration; after n iterations the interval width is $(b - a)/2^n$, thus if tol is target width, required iterations satisfy

$$n \geq \left\lceil \log_2 \frac{b-a}{\mathrm{tol}} \right\rceil . \tag{1.3}$$

1.5 Example 1: $f(x) = x^3 - x - 2$ on $[1, 2]$

Initial interval	$[1, 2]$
Iterations performed	40
Final midpoint	$x_n = 1.521379706805$
Absolute error	$\lvert x_n - x^* \rvert = 8.515e - 13$

1.6 Example 2: $f(x) = \sin x - 0.5$ on $[0, 2]$ (root at $\pi/6$)

Initial interval	$[0, 2]$
Iterations performed	41
Final midpoint	$x_n = 0.523598775598$
Absolute error	$\lvert x_n - x^* \rvert = 5.513e - 13$

1.7 Newton–Raphson Method

1.7.1 Method and Local Error Relation (brief)

Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1.4}$$

Assuming f is sufficiently smooth and $f'(x^*) \neq 0$, a Taylor expansion yields the local quadratic error relation

$$e_{n+1} \approx -\frac{f''(x^*)}{2f'(x^*)} e_n^2, \tag{1.5}$$

showing (quadratic) convergence when close to the root.

1.8 Example 1: $f(x) = x^3 - x - 2$, $x_0 = 1.5$

Initial guess	$x_0 = 1.5$
Iterations performed	4
Final approximation	$x_n = 1.521379706805$
Absolute error	$\lvert x_n - x^* \rvert = 0.000e + 00$
Observed order (estimate)	2.000

1.9 Example 2: $f(x) = x^2 - 2$ (compute $\sqrt{2}$), $x_0 = 1$

Initial guess	$x_0 = 1$
Iterations performed	6
Final approximation	$x_n = 1.414213562373$
Absolute error	$\lvert x_n - x^* \rvert = 2.220e - 16$
Observed order (estimate)	N/A

1.10 Secant Method

1.10.1 Method and Convergence (brief)

The secant method uses the finite-difference approximation to the derivative:

$$x_{n+1} = x_n - f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}. \tag{1.6}$$

Its convergence order is superlinear with $p \approx 1.618$ (the golden ratio) under suitable conditions.

1.11 Example 1: $f(x) = x^3 - x - 2$, $x_0 = 1$, $x_1 = 2$

Initial guesses	$x_0 = 1,\ x_1 = 2$
Iterations performed	8
Final approximation	$x_n = 1.521379706805$
Absolute error	$\lvert x_n - x^* \rvert = 0.000e + 00$
Observed order (estimate)	1.613

1.12 Example 2: $f(x) = \cos x - x$, $x_0 = 0$, $x_1 = 1$

Initial guesses	$x_0 = 0,\ x_1 = 1$
Iterations performed	6
Final approximation	$x_n = 0.739085133215$
Absolute error	$\lvert x_n - x^* \rvert = 0.000e + 00$
Observed order (estimate)	1.595

1.13 Comparison Summary and Recommendations

Method	Derivative?	Typical Order	Iters (example)	Final abs. error (example)
Fixed Point	No	Linear (—g'—)	69, 14	3.063e-13, 3.531e-14
Bisection	No	Linear	40, 41	8.515e-13, 5.513e-13
Newton	Yes	Quadratic	4, 6	0.000e+00, 2.220e-16
Secant	No	≈1.618	8, 6	0.000e+00, 0.000e+00

1.14 Discussion

The numerical examples illustrate typical behavior: Newton converges extremely quickly when derivative information is available; Secant approaches Newton speed without requiring the derivative; Bisection is robust but requires more iterations; Fixed point depends strongly on the chosen transformation g. Hybrid strategies (e.g., use Bisection to bracket, then Secant/Newton to accelerate) are recommended in practice.