

# Chapter 1

## Vector Analysis

### Exercise Series N° 1: Vector Spaces and Operations

#### Exercise 1.1 (Vector Space Verification)

Verify if the following sets with given operations form vector spaces over  $\mathbb{R}$ :

1.  $V = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$  with standard operations
2.  $W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$  with standard operations
3.  $U = \{f : \mathbb{R} \rightarrow \mathbb{R} : f(0) = 1\}$  with pointwise addition and scalar multiplication

#### Solution: Example 1

$$V = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$$

1. **Closure under addition:** Take  $(x_1, y_1), (x_2, y_2) \in V$  with  $x_1 \geq 0, x_2 \geq 0$ . Their sum is  $(x_1 + x_2, y_1 + y_2)$ . Since  $x_1 + x_2 \geq 0$ , closure holds.
2. **Closure under scalar multiplication:** Take  $(x, y) \in V$  with  $x \geq 0$  and  $\lambda \in \mathbb{R}$ . The product is  $(\lambda x, \lambda y)$ . But if  $\lambda < 0$ , then  $\lambda x \leq 0$ , which may violate  $x \geq 0$  condition. For example,  $\lambda = -1$  gives  $(-x, -y) \notin V$  when  $x > 0$ .
3. **Conclusion:**  $V$  is **not** a vector space because it fails closure under scalar multiplication with negative scalars.

#### Solution: Example 2

$$W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$$

1. **Contains zero vector:**  $(0, 0, 0)$  satisfies  $0 + 0 + 0 = 0$ , so  $0_W \in W$ .
2. **Closure under addition:** Take  $(x_1, y_1, z_1), (x_2, y_2, z_2) \in W$  with  $x_1 + y_1 + z_1 = 0$  and  $x_2 + y_2 + z_2 = 0$ . Their sum is  $(x_1 + x_2, y_1 + y_2, z_1 + z_2)$  and  $(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = 0$ , so closure holds.
3. **Closure under scalar multiplication:** Take  $(x, y, z) \in W$  with  $x + y + z = 0$  and  $\lambda \in \mathbb{R}$ . The product is  $(\lambda x, \lambda y, \lambda z)$  and  $\lambda x + \lambda y + \lambda z = \lambda(x + y + z) = 0$ , so closure holds.
4. **Conclusion:**  $W$  is a vector space (it's a plane through the origin).

**Solution: Example 3**

$$U = \{f : \mathbb{R} \rightarrow \mathbb{R} : f(0) = 1\}$$

1. **Contains zero function:** The zero function  $f(x) = 0$  satisfies  $f(0) = 0 \neq 1$ , so  $0_U \notin U$ .
2. **Closure under addition:** Take  $f, g \in U$  with  $f(0) = 1, g(0) = 1$ . Their sum  $(f + g)(0) = f(0) + g(0) = 2 \neq 1$ , so not closed under addition.
3. **Conclusion:**  $U$  is not a vector space.

**Exercise 1.2** (Linear Combinations and Span)

For the vectors in  $\mathbb{R}^3$ :  $\vec{v}_1 = (1, 2, 3)$ ,  $\vec{v}_2 = (2, 5, 7)$ ,  $\vec{v}_3 = (1, 3, 5)$

1. Determine if  $\vec{w} = (4, 11, 18)$  is in the span of  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$
2. Find the general form of vectors in  $\text{span}\{\vec{v}_1, \vec{v}_2\}$
3. Express  $\vec{v}_3$  as a linear combination of  $\vec{v}_1$  and  $\vec{v}_2$  if possible

**Solution: Example 1**

Is  $\vec{w} = (4, 11, 18) \in \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ ?

1. Solve  $a(1, 2, 3) + b(2, 5, 7) + c(1, 3, 5) = (4, 11, 18)$
2. System:

$$a + 2b + c = 4$$

$$2a + 5b + 3c = 11$$

$$3a + 7b + 5c = 18$$

3. Solving: From first equation,  $c = 4 - a - 2b$ . Substitute into second:  $2a + 5b + 3(4 - a - 2b) = 11 \Rightarrow -a - b = -1 \Rightarrow a + b = 1$
4. Substitute into third:  $3a + 7b + 5(4 - a - 2b) = 18 \Rightarrow -2a - 3b = -2 \Rightarrow 2a + 3b = 2$
5. Solve  $a + b = 1$  and  $2a + 3b = 2$ :  $a = 1, b = 0$ , then  $c = 3$
6. Solution exists:  $\vec{w} = 1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + 3 \cdot \vec{v}_3$ , so  $\vec{w} \in \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

**Solution: Example 2**

Find  $\text{span}\{\vec{v}_1, \vec{v}_2\}$

1. General linear combination:  $a(1, 2, 3) + b(2, 5, 7) = (a + 2b, 2a + 5b, 3a + 7b)$
2. This represents all vectors of the form  $(x, y, z)$  where  $x = a + 2b, y = 2a + 5b, z = 3a + 7b$
3. Eliminating parameters: From  $x = a + 2b$  and  $y = 2a + 5b$ , solve for  $a, b$ :

$$a = 5x - 2y$$

$$b = y - 2x$$

4. Substitute into  $z = 3a + 7b = 3(5x - 2y) + 7(y - 2x) = x + y$
5. So  $\text{span}\{\vec{v}_1, \vec{v}_2\} = \{(x, y, z) \in \mathbb{R}^3 : z = x + y\}$

**Solution: Example 3**

Express  $\vec{v}_3$  as combination of  $\vec{v}_1, \vec{v}_2$

1. Solve  $a(1, 2, 3) + b(2, 5, 7) = (1, 3, 5)$
2. System:

$$a + 2b = 1$$

$$2a + 5b = 3$$

$$3a + 7b = 5$$

3. From first two equations:  $a = 1, b = 0$  works for first two but  $3(1) + 7(0) = 3 \neq 5$
4. No solution exists, so  $\vec{v}_3$  is not in the span of  $\{\vec{v}_1, \vec{v}_2\}$

**Exercise 1.3** (Linear Independence Testing)

Determine if the following sets are linearly independent:

1.  $\{(1, 2, 3), (2, 5, 7), (1, 3, 5)\}$  in  $\mathbb{R}^3$
2.  $\{\sin x, \cos x, e^x\}$  in  $\mathcal{F}(\mathbb{R}, \mathbb{R})$
3.  $\{1, x, x^2, x^3\}$  in  $\mathcal{P}_3(\mathbb{R})$

**Solution: Example 1**

$$\{(1, 2, 3), (2, 5, 7), (1, 3, 5)\} \text{ in } \mathbb{R}^3$$

1. Set up equation:  $a(1, 2, 3) + b(2, 5, 7) + c(1, 3, 5) = (0, 0, 0)$
2. System:

$$a + 2b + c = 0$$

$$2a + 5b + 3c = 0$$

$$3a + 7b + 5c = 0$$

3. Matrix form: 
$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 3 & 7 & 5 \end{bmatrix}$$

4. Determinant:  $1(25 - 21) - 2(10 - 9) + 1(14 - 15) = 4 - 2 - 1 = 1 \neq 0$

5. Only trivial solution  $a = b = c = 0$ , so vectors are **linearly independent**

**Solution: Example 2**

$$\{\sin x, \cos x, e^x\} \text{ in } \mathcal{F}(\mathbb{R}, \mathbb{R})$$

1. Set up equation:  $a \sin x + b \cos x + ce^x = 0$  for all  $x \in \mathbb{R}$
2. Test specific values:

- $x = 0$ :  $a \cdot 0 + b \cdot 1 + c \cdot 1 = b + c = 0$

- $x = \frac{\pi}{2}$ :  $a \cdot 1 + b \cdot 0 + c \cdot e^{\pi/2} = a + ce^{\pi/2} = 0$

- $x = \pi$ :  $a \cdot 0 + b \cdot (-1) + c \cdot e^\pi = -b + ce^\pi = 0$

- Solve system: From first and third,  $b = -c$  and  $-(-c) + ce^\pi = 0 \Rightarrow c(1 + e^\pi) = 0 \Rightarrow c = 0$ , then  $b = 0$ , then  $a = 0$
- Only trivial solution, so functions are **linearly independent**

**Solution: Example 3**

$$\{1, x, x^2, x^3\} \text{ in } \mathcal{P}_3(\mathbb{R})$$

- Set up equation:  $a \cdot 1 + b \cdot x + c \cdot x^2 + d \cdot x^3 = 0$  for all  $x \in \mathbb{R}$
- This is a polynomial identity, so coefficients must all be zero
- Therefore  $a = b = c = d = 0$  is the only solution
- So the set is **linearly independent** (it's the standard basis for  $\mathcal{P}_3(\mathbb{R})$ )

**Exercise 1.4** (Basis and Dimension)

For each case, determine if the set forms a basis and find the dimension:

- $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$  for  $\mathbb{R}^3$
- $\{1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3\}$  for  $\mathcal{P}_3(\mathbb{R})$
- $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$  for  $M_{2 \times 2}(\mathbb{R})$

**Solution: Example 1**

$$\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\} \text{ for } \mathbb{R}^3$$

- Check linear independence: Solve  $a(1, 1, 0) + b(1, 0, 1) + c(0, 1, 1) = (0, 0, 0)$
- System:

$$a + b = 0$$

$$a + c = 0$$

$$b + c = 0$$

- Solution:  $a = b = c = 0$ , so linearly independent
- 3 linearly independent vectors in  $\mathbb{R}^3$  form a basis

5. Dimension:  $\dim = 3$

**Solution: Example 2**

$$\{1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3\} \text{ for } \mathcal{P}_3(\mathbb{R})$$

1. Check if they span  $\mathcal{P}_3(\mathbb{R})$ : Any polynomial  $ax^3 + bx^2 + cx + d$  can be expressed as:

$$\begin{aligned} d \cdot 1 + (c - d)(1 + x) + (b - c)(1 + x + x^2) + (a - b)(1 + x + x^2 + x^3) \\ = a(x^3 + x^2 + x + 1) + (b - a)(x^2 + x + 1) + (c - b)(x + 1) + (d - c) \cdot 1 \end{aligned}$$

2. Check linear independence: The coordinate matrix relative to standard basis is:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. Determinant = 1  $\neq$  0, so linearly independent

4. Forms a basis, dimension = 4

**Solution: Example 3**

Matrix set for  $M_{2 \times 2}(\mathbb{R})$

1. These are the standard basis matrices

2. Any  $2 \times 2$  matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = aE_{11} + bE_{12} + cE_{21} + dE_{22}$

3. Clearly spans  $M_{2 \times 2}(\mathbb{R})$  and is linearly independent

4. Forms a basis, dimension = 4

**Exercise 1.5** (Coordinate Systems and Transformations)

Perform the following coordinate transformations:

1. Convert  $P(2, 2\sqrt{3})$  from Cartesian to polar coordinates
2. Convert  $Q(4, \frac{\pi}{3}, 5)$  from cylindrical to Cartesian coordinates
3. Convert  $R(2, \frac{\pi}{4}, \frac{\pi}{6})$  from spherical to Cartesian coordinates

**Solution: Example 1** $P(2, 2\sqrt{3})$  Cartesian to Polar

1.  $r = \sqrt{x^2 + y^2} = \sqrt{4 + 12} = \sqrt{16} = 4$
2.  $\theta = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{2\sqrt{3}}{2}\right) = \arctan(\sqrt{3}) = \frac{\pi}{3}$
3. Polar coordinates:  $(4, \frac{\pi}{3})$

**Solution: Example 2** $Q(4, \frac{\pi}{3}, 5)$  Cylindrical to Cartesian

1.  $x = r \cos \theta = 4 \cos\left(\frac{\pi}{3}\right) = 4 \cdot \frac{1}{2} = 2$
2.  $y = r \sin \theta = 4 \sin\left(\frac{\pi}{3}\right) = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$
3.  $z = 5$
4. Cartesian coordinates:  $(2, 2\sqrt{3}, 5)$

**Solution: Example 3** $R(2, \frac{\pi}{4}, \frac{\pi}{6})$  Spherical to Cartesian

1. Spherical coordinates:  $(\rho, \theta, \phi) = (2, \frac{\pi}{4}, \frac{\pi}{6})$
2.  $x = \rho \sin \phi \cos \theta = 2 \sin\left(\frac{\pi}{6}\right) \cos\left(\frac{\pi}{4}\right) = 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$
3.  $y = \rho \sin \phi \sin \theta = 2 \sin\left(\frac{\pi}{6}\right) \sin\left(\frac{\pi}{4}\right) = 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$
4.  $z = \rho \cos \phi = 2 \cos\left(\frac{\pi}{6}\right) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$
5. Cartesian coordinates:  $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \sqrt{3})$

**Exercise 1.6 (Vector Operations in Different Coordinates)**

Given vectors in different coordinate systems:

1.  $\vec{u} = (3, \frac{\pi}{4})$  in polar,  $\vec{v} = (2, \frac{\pi}{6})$  in polar - find  $\vec{u} + \vec{v}$  in Cartesian
2.  $\vec{a} = (2, \frac{\pi}{3}, 4)$  in cylindrical,  $\vec{b} = (1, \frac{\pi}{6}, -2)$  in cylindrical - find dot product
3.  $\vec{p} = (3, \frac{\pi}{4}, \frac{\pi}{3})$  in spherical, convert to Cartesian and find magnitude

**Solution: Example 1** $\vec{u} + \vec{v}$  in polar coordinates

1. Convert to Cartesian:

$$\vec{u} = \left( 3 \cos \frac{\pi}{4}, 3 \sin \frac{\pi}{4} \right) = \left( \frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2} \right)$$

$$\vec{v} = \left( 2 \cos \frac{\pi}{6}, 2 \sin \frac{\pi}{6} \right) = (\sqrt{3}, 1)$$

2. Sum in Cartesian:  $\left( \frac{3\sqrt{2}}{2} + \sqrt{3}, \frac{3\sqrt{2}}{2} + 1 \right)$

3. Convert back to polar:

$$r = \sqrt{\left( \frac{3\sqrt{2}}{2} + \sqrt{3} \right)^2 + \left( \frac{3\sqrt{2}}{2} + 1 \right)^2}$$

$$\theta = \arctan \left( \frac{\frac{3\sqrt{2}}{2} + 1}{\frac{3\sqrt{2}}{2} + \sqrt{3}} \right)$$

**Solution: Example 2**

$\vec{a} \cdot \vec{b}$  in cylindrical coordinates

1. Convert to Cartesian:

$$\vec{a} = \left( 2 \cos \frac{\pi}{3}, 2 \sin \frac{\pi}{3}, 4 \right) = (1, \sqrt{3}, 4)$$

$$\vec{b} = \left( 1 \cos \frac{\pi}{6}, 1 \sin \frac{\pi}{6}, -2 \right) = \left( \frac{\sqrt{3}}{2}, \frac{1}{2}, -2 \right)$$

2. Dot product:  $1 \cdot \frac{\sqrt{3}}{2} + \sqrt{3} \cdot \frac{1}{2} + 4 \cdot (-2) = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} - 8 = \sqrt{3} - 8$

**Solution: Example 3**

$\vec{p}$  spherical to Cartesian and magnitude

1. Convert:  $(\rho, \theta, \phi) = \left( 3, \frac{\pi}{4}, \frac{\pi}{3} \right)$

$$x = 3 \sin \frac{\pi}{3} \cos \frac{\pi}{4} = 3 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{3\sqrt{6}}{4}$$

$$y = 3 \sin \frac{\pi}{3} \sin \frac{\pi}{4} = 3 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{3\sqrt{6}}{4}$$

$$z = 3 \cos \frac{\pi}{3} = 3 \cdot \frac{1}{2} = \frac{3}{2}$$

2. Magnitude:  $|\vec{p}| = \rho = 3$  (by definition in spherical coordinates)

3. Verify:  $\sqrt{x^2 + y^2 + z^2} = \sqrt{\frac{54}{16} + \frac{54}{16} + \frac{9}{4}} = \sqrt{\frac{108}{16} + \frac{36}{16}} = \sqrt{\frac{144}{16}} = 3$

**Exercise 1.7** (Subspaces and Direct Sums)

For the following subspaces of  $\mathbb{R}^3$ :

1.  $U = \{(x, y, z) : x + y + z = 0\}$ ,  $V = \{(x, y, z) : x = y = z\}$  - find  $U \cap V$  and  $U + V$
2.  $W = \text{span}\{(1, 2, 3), (2, 5, 7)\}$ ,  $Z = \text{span}\{(1, 0, 1), (0, 1, 2)\}$  - find basis for  $W \cap Z$
3. Show that  $\mathbb{R}^3 = U \oplus W$  where  $U = \{(x, y, 0)\}$ ,  $W = \{(0, 0, z)\}$

**Solution: Example 1**

$$U \cap V \text{ and } U + V$$

1.  $U \cap V$ : Vectors satisfying both  $x + y + z = 0$  and  $x = y = z$
2. Let  $x = y = z = t$ , then  $3t = 0 \Rightarrow t = 0$ , so  $U \cap V = \{(0, 0, 0)\}$
3.  $U + V$ : Any vector in  $\mathbb{R}^3$  can be written as sum of vector from  $U$  and vector from  $V$
4. For  $(a, b, c) \in \mathbb{R}^3$ , want  $(a, b, c) = (x, y, z) + (t, t, t)$  with  $x + y + z = 0$
5. Solution:  $t = \frac{a+b+c}{3}$ ,  $x = a - t$ ,  $y = b - t$ ,  $z = c - t$ , and  $x + y + z = 0$
6. So  $U + V = \mathbb{R}^3$

**Solution: Example 2**

$$W \cap Z$$

1.  $W = \text{span}\{(1, 2, 3), (2, 5, 7)\} = \{(x, y, z) : z = x + y\}$  (from previous exercise)
2.  $Z = \text{span}\{(1, 0, 1), (0, 1, 2)\} = \{(x, y, z) : z = x + 2y\}$
3.  $W \cap Z$ : Vectors satisfying both  $z = x + y$  and  $z = x + 2y$
4. Equate:  $x + y = x + 2y \Rightarrow y = 0$ , then  $z = x$
5. So  $W \cap Z = \{(x, 0, x) : x \in \mathbb{R}\} = \text{span}\{(1, 0, 1)\}$
6. Basis:  $\{(1, 0, 1)\}$ , dimension = 1

**Solution: Example 3**

$$\mathbb{R}^3 = U \oplus W$$

1.  $U \cap W$ : Vectors in both  $\{(x, y, 0)\}$  and  $\{(0, 0, z)\}$  must have  $x = y = 0$  and  $z$  arbitrary? Wait, vectors in  $W$  have  $x = y = 0$ , so intersection requires  $x = y = 0$  and  $z$  can be anything, but vectors in  $U$  have  $z = 0$ , so  $U \cap W = \{(0, 0, 0)\}$
2.  $U + W$ : Any  $(a, b, c) = (a, b, 0) + (0, 0, c)$ , so  $U + W = \mathbb{R}^3$
3. Direct sum:  $U \cap W = \{0\}$  and  $U + W = \mathbb{R}^3$ , so  $\mathbb{R}^3 = U \oplus W$

**Exercise 1.8** (Linear Transformations and Coordinates)

Consider the linear transformation  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  defined by  $T(x, y, z) = (x + y, y + z)$ :

1. Find matrix representation relative to standard bases
2. Find kernel and image of  $T$
3. If  $B = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$  is basis for  $\mathbb{R}^3$ , find  $[T]_B$

**Solution: Example 1**

Matrix representation relative to standard bases

1. Standard basis for  $\mathbb{R}^3$ :  $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$
2.  $T(e_1) = (1, 0)$ ,  $T(e_2) = (1, 1)$ ,  $T(e_3) = (0, 1)$
3. Matrix columns:  $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

**Solution: Example 2**

Kernel and image of  $T$

1. Kernel: Solve  $T(x, y, z) = (0, 0)$
2. System:  $x + y = 0$ ,  $y + z = 0$   $y = -x$ ,  $z = -y = x$
3. Kernel =  $\{(x, -x, x) : x \in \mathbb{R}\} = \text{span}\{(1, -1, 1)\}$ , dimension = 1
4. Image:  $T(x, y, z) = (x + y, y + z) = x(1, 0) + y(1, 1) + z(0, 1)$
5. Image =  $\text{span}\{(1, 0), (1, 1), (0, 1)\} = \mathbb{R}^2$  (since these span  $\mathbb{R}^2$ ), dimension = 2

**Solution: Example 3**

Matrix relative to basis  $B$

1. Basis  $B = \{v_1 = (1, 1, 0), v_2 = (1, 0, 1), v_3 = (0, 1, 1)\}$
2.  $T(v_1) = (2, 1), T(v_2) = (1, 1), T(v_3) = (1, 2)$
3. Express in standard basis for  $\mathbb{R}^2$ :  $[T(v_1)] = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, [T(v_2)] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, [T(v_3)] = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
4. So  $[T]_B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

### Exercise 1.9 (Applications to Geometry)

Solve the following geometric problems using vector methods:

1. Find distance between point  $P(1, 2, 3)$  and plane  $x + 2y + 3z = 6$
2. Find angle between lines  $L_1 : (x, y, z) = (1, 2, 3) + t(1, 1, 1)$  and  $L_2 : (x, y, z) = (0, 1, 2) + s(2, -1, 1)$
3. Find volume of parallelepiped determined by vectors  $(1, 2, 3), (2, 5, 7), (1, 3, 5)$

#### Solution: Example 1

Distance from point to plane

1. Plane:  $x + 2y + 3z = 6$ , normal vector  $\vec{n} = (1, 2, 3)$
2. Point  $P(1, 2, 3)$ , choose point  $Q$  on plane, e.g.,  $Q(6, 0, 0)$
3. Vector  $\vec{PQ} = (5, -2, -3)$
4. Distance  $= \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|} = \frac{|5 - 4 - 9|}{\sqrt{1+4+9}} = \frac{8}{\sqrt{14}}$

#### Solution: Example 2

Angle between lines

1. Direction vectors:  $\vec{v}_1 = (1, 1, 1), \vec{v}_2 = (2, -1, 1)$
2. Dot product:  $\vec{v}_1 \cdot \vec{v}_2 = 2 - 1 + 1 = 2$
3. Magnitudes:  $|\vec{v}_1| = \sqrt{3}, |\vec{v}_2| = \sqrt{6}$
4.  $\cos \theta = \frac{2}{\sqrt{3}\sqrt{6}} = \frac{2}{\sqrt{18}} = \frac{\sqrt{2}}{3}$

$$5. \theta = \arccos\left(\frac{\sqrt{2}}{3}\right)$$

**Solution: Example 3**

Volume of parallelepiped

$$1. \text{ Vectors: } \vec{a} = (1, 2, 3), \vec{b} = (2, 5, 7), \vec{c} = (1, 3, 5)$$

$$2. \text{ Scalar triple product: } \vec{a} \cdot (\vec{b} \times \vec{c})$$

$$3. \vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 5 & 7 \\ 1 & 3 & 5 \end{vmatrix} = (25 - 21, -(10 - 7), 6 - 5) = (4, -3, 1)$$

$$4. \vec{a} \cdot (4, -3, 1) = 4 - 6 + 3 = 1$$

$$5. \text{ Volume} = |1| = 1$$

**Exercise 1.10** (Advanced Vector Space Problems)

Solve these advanced problems:

1. Show that  $W = \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) : f'' + f = 0\}$  is a subspace and find its dimension
2. Prove that if  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  is basis for  $\mathbb{R}^3$ , then so is  $\{\vec{v}_1 + \vec{v}_2, \vec{v}_2 + \vec{v}_3, \vec{v}_3 + \vec{v}_1\}$
3. Find basis for the solution space of the system:

$$\begin{cases} x + 2y + 3z = 0 \\ 2x + 5y + 7z = 0 \end{cases}$$

**Solution: Example 1**

$$W = \{f : f'' + f = 0\}$$

1. Check subspace properties:

- Zero function:  $0'' + 0 = 0$ , so  $0 \in W$
- Closed under addition:  $(f + g)'' + (f + g) = f'' + g'' + f + g = (f'' + f) + (g'' + g) = 0$
- Closed under scalar multiplication:  $(\lambda f)'' + \lambda f = \lambda(f'' + f) = 0$

2. So  $W$  is a subspace

3. Solve  $f'' + f = 0$ : Characteristic equation  $r^2 + 1 = 0$ , roots  $r = \pm i$
4. General solution:  $f(x) = A \cos x + B \sin x$
5. Basis:  $\{\cos x, \sin x\}$ , dimension = 2

**Solution: Example 2**

New basis from old basis

1. Let  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  be basis for  $\mathbb{R}^3$
2. Consider  $\{\vec{u}_1 = \vec{v}_1 + \vec{v}_2, \vec{u}_2 = \vec{v}_2 + \vec{v}_3, \vec{u}_3 = \vec{v}_3 + \vec{v}_1\}$
3. Check linear independence: Solve  $a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3 = 0$
4.  $a(\vec{v}_1 + \vec{v}_2) + b(\vec{v}_2 + \vec{v}_3) + c(\vec{v}_3 + \vec{v}_1) = (a + c)\vec{v}_1 + (a + b)\vec{v}_2 + (b + c)\vec{v}_3 = 0$
5. Since  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  linearly independent:

$$a + c = 0$$

$$a + b = 0$$

$$b + c = 0$$

6. Solution:  $a = b = c = 0$ , so  $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$  linearly independent
7. 3 linearly independent vectors in  $\mathbb{R}^3$  form a basis

**Solution: Example 3**

Solution space basis

1. System:

$$x + 2y + 3z = 0$$

$$2x + 5y + 7z = 0$$

2. Augmented matrix:  $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 5 & 7 & 0 \end{bmatrix}$

3. Row reduce:  $R_2 \rightarrow R_2 - 2R_1$ :  $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

4.  $R_1 \rightarrow R_1 - 2R_2$ : 
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

5. Solution:  $x = -z$ ,  $y = -z$ ,  $z$  free

6. General solution:  $(x, y, z) = z(-1, -1, 1)$

7. Basis:  $\{(-1, -1, 1)\}$ , dimension = 1