Differential Equations 2

2.1 *Basics of Differential Equations*

This chapter includes a set of definitions and concepts in differential equations, among the most important of these concepts:

Definition 2.1.1 *A differential equation: is an equality relationship between an independent variable, say x, and a dependent variable, say $y(x)$, with one or more differential derivatives...*

Definition 2.1.2 *Order of a differential equation: is the order of the highest derivative in the equation.*

Definition 2.1.3 *Degree of a differential equation: is the degree or power or exponent of the highest differential coefficient in the equation, provided the equation does not contain coefficients with fractional powers.*
Or it is said to be the highest exponent of the highest order derivative in the equation.

2.1.1 Solving Differential Equations

Definition 2.1.4 *We call the function $y = y(x)$ a solution of the differential equation $F(x, y, y', y'', \ldots, y^{(n)})$ if:*
1- *It is differentiable n times.*
2- *It satisfies the differential equation, i.e.:*

$$F(x, y(x), y'(x), \ldots, y^{(n)}(x)) = 0$$

2.1.2 General Solution and Particular Solution of Differential Equations

Definition 2.1.5 *The general solution of a differential equation of order n is a solution that contains n arbitrary constants and of course satisfies the differential equation.*

Definition 2.1.6 *For any differential equation of order n, we find that its general solution always depends on n arbitrary constants and is written in the form:*

$$F(x, y, c_1, c_2, \ldots, c_n) = 0$$

2.1.3 Initial Conditions and Boundary Conditions

In problems where you are required to verify the solution of an ordinary differential equation, you can also find the arbitrary constants appearing in the general solution of the equation, which is done through the initial conditions given at the beginning.

In the case of a general solution of a second-order differential equation, for example, containing two arbitrary constants, two additional conditions for the equation are needed to determine the constants.

If the two conditions are given at two different points $y(x_1) = y_1$, $y(x_2) = y_2$, then the conditions are boundary conditions, and we call the differential equation along with the boundary conditions a boundary value problem.

2.2 *Definitions in Differential Equations*

Definition 2.2.1 *A differential equation is any equation that contains differentials or derivatives of one or more functions with respect to variables, and it is of the form:*

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0 \qquad (E)$$

Example 2.2.1

$$\frac{dx}{dy}z + ydx = u$$

Differential equations are classified into:

1- Ordinary differential equation: is a differential equation that contains ordinary derivatives or differentials of one or more functions.

Example 2.2.2

$$ydx + xdy = e^z$$

2- Partial differential equation: is a differential equation that contains partial derivatives or differentials of one or more functions.

Example 2.2.3

$$\frac{\partial x}{\partial y} = zx$$

3- Linear ordinary differential equation: is an equation that is linear with respect to the function or functions and their derivatives and does not contain their products.

4- Linear partial differential equation: is an equation that is linear with respect to the partial derivatives of the function or functions present.

Remark 2.2.1 1- The order of the equation is the order of the highest derivative present in it.
2- The differential equation can be transformed from one form to another to facilitate its solution.

2.3 *First Order Differential Equations*

Definition 2.3.1 *A first-order differential equation is a relationship between a function (considered unknown) y and its first derivative and the variable x of y.*

$$\frac{dy}{dx} = F(x, y)$$

or

$$M(x, y)d(x) + N(x, y)d(y)$$

To solve such equations, we follow the following methods:

2.3.1 Separation of Variables Method

If the equation can be put in the form:

$$f(x)dx + g(y)dy = 0$$

where $f(x)$ is a function in x only and $g(y)$ is a function in y only, then the separation of variables has been achieved. To solve the equation after separation, we use direct integration, and the solution becomes:

$$\int f(x)dx + \int g(y)dy = c$$

where c is an arbitrary constant, and this solution is called the general solution. The arbitrary constant can be put in any form according to the requirements of simplifying the form of the general solution.

And if an initial condition is known, we can eliminate the arbitrary constant and the resulting solution becomes a particular solution.

Example 2.3.1 Solve the following differential equation:

$$xy^2dx + (1 - x^2)dy = 0$$

Solution: We divide both sides of the equation by $y^2(1 - x^2)$ to obtain:

$$\frac{xdx}{1 - x^2} + \frac{dy}{y^2} = 0$$

which is a differential equation that can be separated by variables, and its solution method is as follows:
By integrating both sides:

$$\begin{aligned} \int \frac{xdx}{1 - x^2} + \int \frac{dy}{y^2} &= 0 \Rightarrow -\frac{1}{2}\ln\left(x^2 - 1\right) - \frac{1}{y} = c \\ &\Rightarrow \ln\left(x^2 - 1\right)^{-\frac{1}{2}} - \frac{1}{y} = c \\ &\Rightarrow \frac{1}{y} = \ln\left(x^2 - 1\right)^{-\frac{1}{2}} - c \end{aligned}$$

And thus the solution of the differential equation is:

$$y = \left(\ln\left(x^2 - 1\right)^{-\frac{1}{2}} - c\right)^{-1}$$

2.3.2 Exact Differential Equations

If the differential equation:

$$P(x, y)dx + Q(x, y)dy = 0$$

is exact, then there exists a function $f(x, y)$ such that:

$$P(x, y)dx + Q(x, y)dy = df$$

That is:

$$\frac{\partial f}{\partial x} = P(x, y) \tag{1}$$

and

$$\frac{\partial f}{\partial y} = Q(x, y) \tag{2}$$

By differentiating equation (1) with respect to y and equation (2) with respect to x, we find:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial P}{\partial y} \text{ and } \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial Q}{\partial x}$$

And for the equation to be exact, it is necessary to satisfy the following condition:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

which is satisfied because:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

To solve the exact equation, there are a few steps we follow:

- We assume a function that satisfies:

$$df\,(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

Then its solution is $f(x,y) = c$ where c is a constant. And it satisfies:

$$\frac{\partial f}{\partial x} = P \text{ and } \frac{\partial f}{\partial y} = Q$$

By integration we find:

$$f\,(x,y) = \int P\,(x,y)\,dx + \psi\,(y) \tag{3}$$

where $\psi\,(y)$ is constant with respect to x.

- We differentiate both sides of (3) partially with respect to y and using the equation $\frac{\partial f}{\partial y} = Q$, we find:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int P\,(x,y)\,dx + \psi'\,(y) = Q\,(x,y)$$

That is:

$$\psi'\,(y) = Q\,(x,y) - \frac{\partial}{\partial y}\int P\,(x,y)\,dx$$

We note that the right side of the last equation is always a function of y only. (Why?)

- By integrating both sides of the last equation with respect to y, we deduce the form of the function $\psi'(y)$ where:

$$\psi(y) = \int Q(x,y)\,dy - \int \left(\frac{\partial}{\partial y}\int P(x,y)\,dx\right)dy$$

And by substitution in equation (3), we obtain the solution of the exact differential equation, and it is in the form:

$$\int P(x,y)\,dx + \int Q(x,y)\,dy - \int \left(\frac{\partial}{\partial y}\int P(x,y)\,dx\right)dy = c$$

Example 2.3.2 Find the solution of the equation:

$$(6x^2 + 4xy + y^2)dx + (2x^2 + 2xy - 3y^2)dy = 0$$

Solution:

Let:

$$\begin{aligned} \frac{\partial P}{\partial y} &= \frac{d(6x^2 + 4xy + y^2)}{dy} = 4x + 2y \\ \frac{\partial Q}{\partial x} &= \frac{d(2x^2 + 2xy - 3y^2)}{dx} = 4x + 2y \end{aligned}$$

This means that the equation is exact. To solve the equation, we follow the following: let

$$\begin{aligned} P(x,y) &= \frac{\partial f}{\partial x} = 6x^2 + 4xy + y^2 \\ Q(x,y) &= \frac{\partial f}{\partial y} = 2x^2 + 2xy - 3y^2 \end{aligned}$$

By integration we find:

$$\begin{aligned} \int P(x,y)\,dx &= 2x^3 + 2x^2y + xy^2 \\ \int Q(x,y)\,dy &= 2x^2y + xy^2 - y^3 \end{aligned}$$

By differentiation we find:

$$\frac{\partial}{\partial y}\int P(x,y)\,dx = 2x^2 + 2xy$$

And by integration we find:

$$\int \left(\frac{\partial}{\partial y} \int P\left(x, y\right) dx \right) dy = 2x^2 y + xy^2$$

Applying the formula, we obtain:

$$2x^3 + 2x^2y + xy^2 + 2x^2y + xy^2 - y^3 - \left(2x^2y + xy^2\right) = c$$

$$\Leftrightarrow c = f(x, y) = 2x^3 + 2x^2y + xy^2 - y^3.$$

2.3.3 Homogeneous Differential Equations

This type of differential equations, which are not originally separable by variables, become separable after a variable transformation.

These equations can be written in the form:

$$\frac{dy}{dx} = F\left(\frac{x}{y}\right)$$

This type of equation becomes separable by setting $v = y/x$, we find that:

$$y = vx \Rightarrow \frac{dy}{dx} = x\frac{dy}{dx} + v.$$

And by substitution in the original equation, we obtain a first-order differential equation that can be separated by variables.

Example 2.3.3 Find the solution of the following equation:

$$y' = \frac{x^2 + y^2}{xy}$$

By setting $F\left(x, y\right) = \frac{x^2+y^2}{xy}$, we can write the following:

$$F\left(x, y\right) = \frac{x}{y} + \frac{y}{x}$$

And by setting $v = y/x$, we deduce that the differential equation is homogeneous. By substitution in the original differential equation, we find that:

$$dy = \left(\frac{1}{v} + v\right) dx$$

But: $y = xv$

By differentiating both sides, it becomes:

$$dy = xdv + vdx$$

And by substituting for dy, we obtain the following relation:

$$xdv + vdx = \left(\frac{1}{v} + v\right) dx \Rightarrow xdv = \frac{1}{v}dx \Rightarrow vdv = \frac{dx}{x}$$

By integrating both sides, we find:

$$\int vdv = \int \frac{dx}{x} + c \Rightarrow \frac{1}{2}v^2 = \ln x + c$$

And the equation can be written in the form:

$$y^2 = 2x^2 \ln x + 2x^2 c.$$

Definition 2.3.2 *Differential equations that are written in the form:*

$$N(x, y) \frac{dy}{dx} = M(x, y)$$

where N, M are homogeneous functions of the same degree, are called homogeneous differential equations.
And they can be written in the form:

$$\frac{dy}{dx} = F\left(\frac{x}{y}\right)$$

And thus after variable transformation, they become separable by variables.

Definition 2.3.3 *We say that the function $g(x, y)$ defined for all values of (x, y) is homogeneous of degree n if:*

$$g(tx, ty) = t^n g(x, y)$$

for all values of (x, y).

Example 2.3.4 Determine if the following equation is homogeneous and then find its solution?

$$xy' - y = xe^{x/y}$$

Solution: We can write:

$$xy' - y = xe^{x/y} \Rightarrow xy' = y + xe^{x/y}$$

Let:

$$N(x,y) = x \text{ and } M(x,y) = y + xe^{x/y}$$

We find:

$$N(tx,ty) = tx = tN(x,y)$$

and

$$M(tx,ty) = ty + txe^{tx/ty} = t\left(y + xe^{x/y}\right) = tM(x,y)$$

Therefore, the functions N, M are homogeneous functions of the first degree, and they can be written in the form:

$$\frac{dy}{dx} = F\left(\frac{x}{y}\right)$$

The solution method is as follows: By dividing both sides of the equation by x, the equation becomes:

$$y' - \frac{y}{x} = e^{y/x} \Rightarrow y' = \frac{y}{x} + e^{y/x}$$

By setting $v = y/x$, we find:

$$y = vx \Rightarrow y' = x\frac{dv}{dx} + v$$

Substitute in the given differential equation to obtain:

$$x\frac{dv}{dx} + v = v + e^{v} \Rightarrow x\frac{dv}{dx} = e^{v} \Rightarrow \frac{dx}{x} = e^{-v}dv$$

By integrating both sides, we obtain:

$$\int e^{-v}dv = \int \frac{dx}{x} + c \Longrightarrow -e^{-v} = \ln x + c$$

By applying ln to both sides:

$$\ln\left(-e^{-v}\right) = \ln(\ln x + c) \Longrightarrow v = \ln(\ln x + c)$$

By setting $v = y/x$, the equation becomes:

$$\frac{y}{x} = \ln(\ln x + c) \Longrightarrow y = x\ln(\ln x + c)$$

2.4 *Linear Differential Equations*

Definition 2.4.1 *A differential equation is linear if the dependent variable and its derivatives in the equation are of the first degree.*

The general form of a first-order linear differential equation is:

$$\frac{dy}{dx} + yP(x) = Q(x)$$

And it is called linear in y.
As for the linear equation in x, it is in the form:

$$\frac{dx}{dy} + xa(y) = b(y)$$

The general solution of the first-order differential equation of the form:

$$y = e^{-I(x)} \left(\int e^{I(t)} Q(t)\, dt + c \right)$$

where:

$$I(x) = \int P(x)\, dx$$

and c is a constant.

Example 2.4.1 Find the general solution of the following differential equation:

$$(y + y^2)dx - (y^2 + 2xy + x)dy = 0$$

Solution: The equation is linear in x, as it can be put in the following form:

$$\frac{dx}{dy} + xa(y) = b(y)$$

By dividing both sides of the equation by $dy(y + y^2)$, we find:

$$\frac{dx}{dy} - \frac{y^2 + 2xy + x}{y + y^2} = 0$$

That is:

$$\frac{dx}{dy} - \frac{y^2}{y + y^2} - \frac{2xy + x}{y + y^2} = 0 \Longrightarrow \frac{dx}{dy} - \frac{2y + 1}{y + y^2} x = \frac{y^2}{y + y^2}$$

By comparing the resulting equation with the first equation, we find:

$$b(y) = \frac{y^2}{y + y^2}, \quad a(y) = -\frac{2y + 1}{y + y^2}$$

Therefore:

$$I(y) = e^{-\int \frac{2y+1}{y+y^2} dy} = e^{\ln\left(\frac{1}{y+y^2}\right)} = e^{-\ln\left(y+y^2\right)} = \frac{1}{y + y^2}$$

And:

$$\int I(y)\,b(y)\,dy = \int \frac{1}{y+y^2}\frac{y}{y+1}dy = \int \frac{1}{(y+1)^2}dy = -\frac{1}{y+1}$$

The solution of the equation is:

$$I(y)\,x = \int I(y)\,b(y)\,dy + c$$

$$\frac{1}{y+y^2}x = -\frac{1}{y+1} + c$$

That is:

$$x = -y + c(y^2 + y), c \in \mathbb{R}$$

This is the general solution of the differential equation.

2.5 *Second Order Differential Equations*

Let the following illustrative example that explains how to find solutions of a second-order differential equation.

Example 2.5.1 It is required to find solutions of the following equation:

$$x^2y'' + xy' + y = 2$$

You can solve this equation in several ways, including the method we mentioned earlier, but this equation differs from the equation:

$$y'' + ay' + by = 0$$

in that the coefficients of this are functions of x.

But the other type is linear equations with constant coefficients:

$$y'' + ay' + by = Q(x) \qquad (II)$$

And their general solution is:

$$y = C_1e^{r_1x} + C_2e^{r_2x} + \text{ particular solution}$$

Meaning we solve it as if it were a homogeneous differential equation and then find the particular integral that expresses the function on the right side.

We start first by solving a homogeneous differential equation, let it be:

$$y'' + 3y' - 4y = 0$$

In this case: we assume that:

$$y = Ce^{rx}$$

where r is some real number (constant) we will explain later how to obtain it. Now we find the first and second derivatives in the above differential equation:

$$y' = re^{rx} \text{ and } y'' = r^2e^{rx}$$

By substitution in the equation, we find:

$$y'' + 3y' - 4y = 0$$

$$r^2e^{rx} + 3re^{rx} - 4e^{rx} = 0$$

By taking e^{rx} as a common factor, we find:

$$e^{rx}[r^2 + 3r - 4] = 0$$

And then either $e^{rx} = 0$ which is impossible, so we take the second solution:

$$r^2 + 3r - 4 = 0 \Longrightarrow (r + 4)(r - 1) = 0.$$

We find $r = 1$ or $r = -4$.
And accordingly, all possible solutions of the previous equation are:

$$y_1 = C_1e^x \text{ or } y_2 = C_2e^{-4x}$$

where C_1 and C_2 are constants.
It can be proven that any linear combination of the two solutions also forms a solution of the equation. Therefore, the general solution of the equation is of the form:

$$y = C_1e^x + C_2e^{-4x}$$

And in general, the general solution of the differential equation $y'' + ay' + by = 0$ is of the form:

$$y = C_1e^{r_1x} + C_2e^{r_2x}$$

where r_1 and r_2 are roots of the characteristic function:

$$r^2 + ar + b = 0$$

Now we come to non-homogeneous equations, i.e., those of the form:

$$y'' + ay' + by = Q(x) \qquad (II)$$

And their general solution is:

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \text{ particular solution}$$

Let now the non-homogeneous equation:

$$y'' + 3y' - 4y = x^2$$

We notice it is the same function as before with x^2 instead of zero. This type of equations is solved as if it were: $y'' + 3y' - 4y = 0$ homogeneous.
And its general solution as we mentioned earlier:

$$y = C_1 e^{x} + C_2 e^{-4x}$$

This solution is a partial solution of this differential equation, now we look for the solution that confirms to us the validity of:

$$y'' + 3y' - 4y = x^2$$

Now we notice that the right side is a polynomial function of the second degree, so we assume that its particular solution is a function of the second degree and its general form is of the form:

$$y = ax^2 + bx + c$$

Then $y' = 2ax + b$ and $y'' = 2a$. By substitution in the equation, we find:

$$y'' + 3y' - 4y = x^2 \Longrightarrow 2a + 3(2ax + b) - 4(ax^2 + bx + c) = x^2$$

We arrange this equation from the highest power to the lowest power, considering that a, b, c are constants.

$$2a + 6ax + 3b - 4a^2 - 4bx - 4c - x^2 = 0$$

$$(-4a - 1)x^2 + (6a - 4b)x + (2a + 3b - 4c) = 0$$

For this equation to be true, we require that each of these factors equals zero, i.e.:

$$4a - 1 = 0 \Longrightarrow a = -1/4$$

$$6a - 4b = 0 \Longrightarrow b = -3/8$$

And finally $2a + 3b - 4c = 0$ gives us $c = -13/32$. Therefore, the general solution of this non-homogeneous equation is:

$$y = C_1 e^x + C_2 e^{-4x} - (1/4)x^2 - (3/8)x - (13/32)$$

What do we do if the function is $Q(x) = A\sin(x)$ for example and not x^2? where A is a constant:

Here we assume that the particular solution is of the following form:

$$y = C\sin x + D\cos x$$

By calculating the first and second derivatives and substituting in the original function, and then we set conditions as we did to find both C and D.

If the function is $Q(x) = ax$, then we assume that the particular solution is an affine function, i.e., of the form $y = Cx + D$.

If the function on the right side is $Q(x) = e^{Ax}$, then we assume: $y = Ce^{Ax}$.

In short, the assumption is of the same type as the function on the right side.

In a similar manner, we move to the case where the factors are another function in x.

$$P(x)y'' + q(x)y' + R(x)y = Q(x) \qquad (III)$$

We continue solving the presented example to understand together the method of solving differential equations of this type.

$$x^2y'' + xy' + y = 2$$

We notice that the right side is a function of x or consider it a constant function in x (and the problem is finished), meaning we deal with a non-homogeneous second-order differential equation.

The Euler-Cauchy method is summarized in one idea:

You can transform the previous equation into another equation of the form:

$$y'' + ay' + by = 2$$

where a, b are constants, but for this method to succeed, we transfer the function from the variable x to another variable t (which is partially similar to the Laplace transform method).

We notice in the previous equation that when writing y', we mean $\frac{dy}{dx}$, and when we write y'', we mean $\frac{d^2y}{d^2x}$, i.e., the second derivative with respect to x.
Now we put a transformation that converts $\frac{dy}{dx}$ to $\frac{dy}{dt}$.
We assume that: $x = e^t$. We differentiate both sides with respect to t:

$$\frac{dx}{dt} = e^t$$

We know that $e^t = x$ and the first derivative is also in terms of x, but we want $\frac{dy}{dt}$ using the following rule:

$$\frac{dy}{dt} = \frac{dx}{dt}.\frac{dy}{dx}$$

But we have $\frac{dx}{dt} = x$, therefore:

$$\frac{dy}{dt} = x\frac{dy}{dx}$$

We differentiate once again with respect to the variable t, we find:

$$\frac{d^2y}{d^2x} = \frac{d}{dt}\left(x\frac{dy}{dx}\right)$$

You can simplify the solution as follows:

$$\frac{d^2y}{d^2x} = \frac{d}{dx}\left(x\frac{dy}{dx}\right).\frac{dx}{dt}$$

That is:

$$\frac{d^2y}{d^2x} = \left(\frac{dy}{dx} + x\frac{d^2y}{d^2x}\right)\frac{dx}{dt}$$

We know that $\frac{dy}{dt} = x$. By substitution:

$$\frac{d^2y}{d^2x} = \left(\frac{dy}{dx} + x\frac{d^2y}{d^2x}\right)x$$

$$\frac{d^2y}{d^2x} = x\frac{dy}{dx} + x^2\frac{d^2y}{d^2x}$$

Take a look at the beginning of the problem, you find that:

$$x\frac{dy}{dx} = \frac{dy}{dt}$$

By substitution:

$$\frac{d^2y}{d^2x} = \frac{dy}{dt} + x^2\frac{d^2y}{d^2x}$$

Therefore:

$$x^2\frac{d^2y}{d^2x} = \frac{d^2y}{d^2x} - \frac{dy}{dt}$$

By substitution in the original equation $x^2y'' + xy' + y = 2$, we find:

$$\frac{d^2y}{d^2x} - \frac{dy}{dt} + \frac{dy}{dt} + y = 2$$

Simplify.

$$\frac{d^2y}{d^2x} + y = 2$$

It has transformed into a non-homogeneous differential equation in the variable t. So you can solve it as we mentioned before.

Its solution is of the form:

$$y = C_1e^{r_1t} + C_2e^{r_2t} + \text{ particular solution}$$

where C_1 and C_2 are constants. r_1 and r_2 are roots of the characteristic equation.

First, we find the partial solution of the above equation by setting:

$$\frac{d^2y}{d^2x} + y = 0$$

The characteristic equation is: $r^2 + 1 = 0$ and hence $r_1 = i$ and $r_2 = -i$ where i is the imaginary unit. Therefore, the solution is of the form:

$$y = C_1e^{it} + C_2e^{-it} + \text{ particular solution}$$

And here we want to simplify this expression (put it in another form which is Euler's formula).

$$C_1e^{it} = C_1\cos(t) + iC_1\sin(t)$$

and

$$C_2e^{-it} = C_2\cos(t) - iC_2\sin(t)$$

By adding the two equations together (taking into account similar terms), we find:

$$y = C_1 e^{it} + C_2 e^{-it} = (C_1 + C_2)\cos(t) + i(C_1 - C_2)\sin(t)$$

And by substitution in the equation $\frac{d^2y}{d^2x} + y = 2$, meaning that the particular solution is equal to 2, the equation becomes:

$$y = A\cos(t) + B\sin(t) + 2$$

Returning to: $x = e^t$, by taking ln of both sides results: $t = \ln(x)$ and finally the form of the equation (in x) is:

$$y = A\cos(\ln(x)) + B\sin(\ln(x)) + 2$$

And this is the general solution of the equation.

Theorem 2.5.1 *Let the differential equation:*

$$y'' + ay' + by = Q(x) \qquad (I)$$

And let Δ be the discriminant of its characteristic equation:

$$r^2 + ar + b = 0$$

1- *If $\Delta > 0$ and r_1 and r_2 are roots of the characteristic equation, then its general solution is:*

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \textit{ particular solution}$$

where C_1 and C_2 are constants.

2- *If $\Delta = 0$ and r is a repeated root of the characteristic equation, then its general solution is:*

$$y = e^{rx}(C_1 + C_2 x) + \textit{ particular solution}$$

where C_1 and C_2 are constants.

3- *If $\Delta < 0$ and $r = \alpha + i\beta$ is a root of the characteristic equation, then its general solution is:*

$$y = e^{\alpha x}(C_1\cos(\beta x) + C_2\sin(\beta x)) + \textit{ particular solution}$$

where C_1 and C_2 are constants.