Chapter 2

Real-valued Functions

A real-valued function of a real variable relates a real value to any number within its domain. This type of numerical function makes it possible, in particular, to formulate a relationship between two physical quantities. It is characterized by its graphical representation in the coordinate plane, and can also be defined by a specific formula, differential equation, or analytical form.

2.1 Numerical Functions

Definition 2.1

Let E and F be two sets and f be a relation from the set E to the set F. We say that f is a function if every element of E is associated with at most one element of F, and we write:

$$\begin{array}{rccc} f: & E & \longrightarrow & F \\ & x & \longmapsto & f(x) = y \end{array}$$

Definition 2.2

We say that f is a numerical function if and only if:

$$\begin{array}{rccc} f: & E \subset \mathbb{R} & \longrightarrow & F \subset \mathbb{R} \\ & x & \longmapsto & f(x) = y \end{array}$$

In other words, f is a numerical function if and only if for every element x in E, its image in F is at most one real number.

Example 2.1 (Linear Function)

The simplest numerical function is the linear function:

$$\begin{array}{rcl} f:\mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x)=ax+b \end{array}$$

where a and b are real constants. Its graph is a straight line with slope a and y-intercept b.

Example 2.2 (Inverse Function)

The inverse function:

$$\begin{array}{rcl} f:\mathbb{R}\setminus\{0\} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{1}{x} \end{array}$$

is defined for all real numbers except zero. Its graph is a hyperbola with two branches.

2.1.1 Domain of Definition

To determine the domain of a numerical function, we need to find the set of numbers for which the function is defined. The domain of a numerical function is defined as follows:

Definition 2.3

Let f be a numerical function. The domain of f, denoted by $\mathrm{Dom}(f)$, is the set of all real numbers x such that $f(x)$ is a well-defined real number:

$$\mathrm{Dom}(f)=\{x\in\mathbb{R}\mid f(x)\in\mathbb{R}\}$$

In other words, the domain of a numerical function is the set of all values for which the function is defined and has a real number output:

$$\begin{array}{rcl} f:\mathrm{Dom}(f)\subset\mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) \end{array}$$

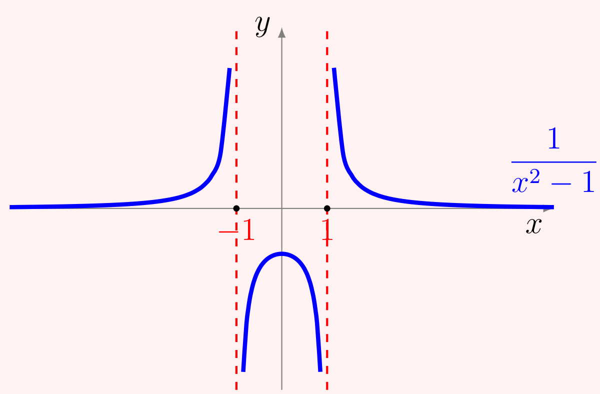
Example 2.3 (Rational Function)

Consider the function:

$$\begin{array}{rcl} f:\mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = \dfrac{1}{(x^2-1)} \end{array}$$

The variable x is in the denominator of the function. Since division by zero is undefined, we must exclude values that make the denominator zero:

$$\begin{aligned} x^2 - 1 = 0 &\iff (x-1)(x+1) = 0 \\ &\iff x = 1 \text{ or } x = -1 \\ &\iff D_f = \mathbb{R} \setminus \{-1, 1\} \\ &\iff D_f =]-\infty, -1[\cup]-1, 1[\cup]1, +\infty[\end{aligned}$$



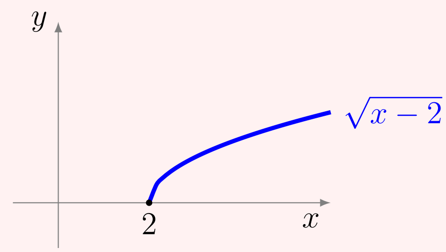
Example 2.4 (Square Root Function)

Consider the function:

$$\begin{array}{rcl} f:\mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = \sqrt{x-2} \end{array}$$

The domain consists of all real numbers x for which the expression under the square root is non-negative:

$$\begin{aligned} x - 2 &\geq 0 \\ x &\geq 2 \\ D_f &= [2, +\infty[\end{aligned}$$



2.1.2 Suggestions for Domain Determination

When finding the domain of a function, consider the following restrictions:

- **Denominators** cannot be zero
- **Expressions under even roots** (square roots, etc.) must be non-negative
- **Logarithmic expressions** must be positive
- **Trigonometric functions** may have periodic restrictions

Example 2.5 (Logarithmic Function)

Consider the function:

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) = \ln(x+3) \end{aligned}$$

The argument of the logarithm must be positive:

$$\begin{aligned} x + 3 &> 0 \\ x &> -3 \\ D_f &=]-3, +\infty[\end{aligned}$$

Example 2.6 (Piecewise Function)

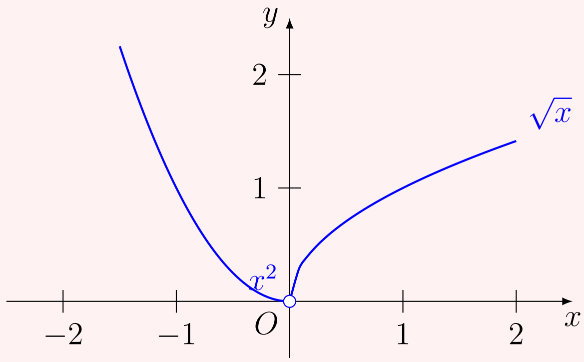
Consider the piecewise-defined function:

$$f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$$

The domain is all real numbers since:

- For $x < 0$, x^2 is defined for all x
- For $x \geq 0$, $\sqrt{x}$ is defined

Thus, $D_f = \mathbb{R}$.



2.1.3 Function Curves and Their Properties

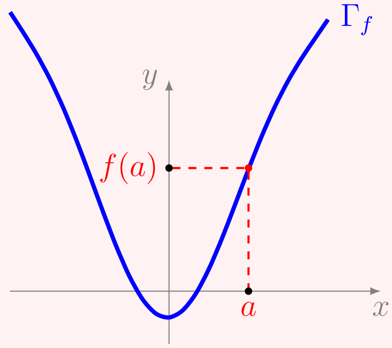
Function curves graphically represent the relationship between inputs and outputs of a function in the Cartesian plane. The curve's shape reveals key properties such as continuity, differentiability, and asymptotic behavior. Important features include intercepts (where the curve crosses the axes), extrema (peaks and valleys), intervals of increase/decrease, and symmetry (even/odd/periodic). For example, a parabola opening upward indicates a quadratic function with a minimum, while oscillating curves suggest periodicity, as seen in trigonometric functions. Discontinuities like jumps or vertical asymptotes highlight domain restrictions. Analyzing these visual characteristics provides immediate insights into the function's behavior, making graphs indispensable tools in calculus and applied mathematics.

Definition 2.4

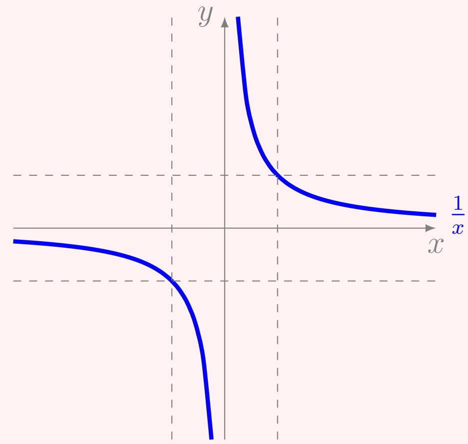
The graph of the function $f : U \to \mathbb{R}$ is the subset Γ_f of $\mathbb{R}^2$ defined as:

$$\Gamma_f = \left\{(x, f(x)) \mid x \in U\right\}.$$

Example 2.7 (Basic Function Graphs)



Graph of $f(x) = \frac{1}{2} + \frac{x^2}{2} + \sin\left(\frac{3(x-1)}{2}\right)$

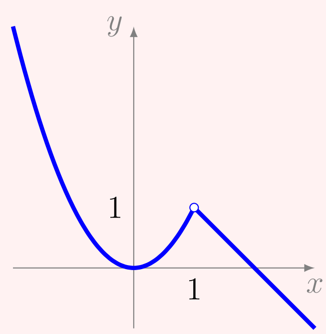


Graph of $f(x) = \frac{1}{x}$ with asymptotes

Example 2.8 (Piecewise Function)

Consider the piecewise function:

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 2 - x & \text{if } x > 1 \end{cases}$$



Graph of a piecewise function with transition at $x = 1$

2.2 Parity and Periodicity

In this section, we will learn how to determine whether a function is even, odd, or neither, using its graph or its definition. The symmetry of the function's curve indicates whether it is odd or even.

2.2.1 Even and Odd Functions

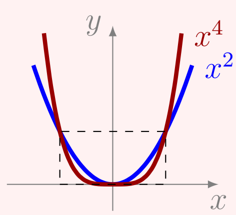
Definition 2.5

A function f is:

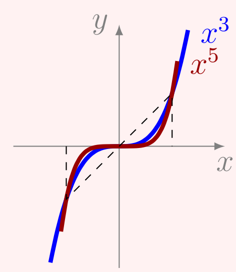
- **Even** if $\forall x \in D_f : f(-x) = f(x)$ (symmetric about y-axis)
- **Odd** if $\forall x \in D_f : f(-x) = -f(x)$ (symmetric about origin)

Example 2.9 (Common Even and Odd Functions)

Function Type	**Example**	**Parity**
Polynomial with even powers	$f(x) = x^4 - 2x^2 + 1$	Even
Polynomial with odd powers	$f(x) = x^3 - x$	Odd
Mixed polynomial	$f(x) = x^3 + x^2$	Neither
Trigonometric	$f(x) = \cos x$	Even
Trigonometric	$f(x) = \sin x$	Odd
Absolute value	$f(x) = \|x\|$	Even



Even functions: x^2 and x^4



Odd functions: x^3 and x^5

If the function f is even, this means that $f(-x) = f(x)$ for all x in the domain of the function. Therefore, if we replace x with $-x$ in the point $M(x_0, f(x_0))$, the point $M'(-x_0, f(-x_0)) = (-x_0, f(x_0))$ is obtained.

Regarding the axis of symmetry, it represents the x-axis, so only the x-coordinates are exchanged. Therefore, we can see that the point $M'(-x_0, f(x_0))$ is the reflection of the point $M(x_0, f(x_0))$ with respect to the axis of symmetry. Thus, the points M and M' are symmetric with respect to the axis of symmetry.

If the function f is odd, this means that $f(-x) = -f(x)$ for all x in the domain of the function. Therefore, if we replace x with $-x$ in the point $M(x_0, f(x_0))$, the point $M'(-x_0, f(-x_0)) =$

$(-x_0, -f(x_0))$ is obtained.

Regarding the origin, it is the point $(0, 0)$ on the coordinate plane. Therefore, we can see that the point $M'(-x_0, f(-x_0))$ is the reflection of the point $M(x_0, f(x_0))$ with respect to the origin. Thus, the points M and M' are symmetric with respect to the origin.

2.2.2 Periodic Functions

Graphically, periodic functions refer to a pattern that is repeated regularly in the Cartesian plane. To fully understand the concept of periodicity, it is important to master the concepts of cycle and period.

Definition 2.6

A function f is **periodic** with period $T > 0$ if:

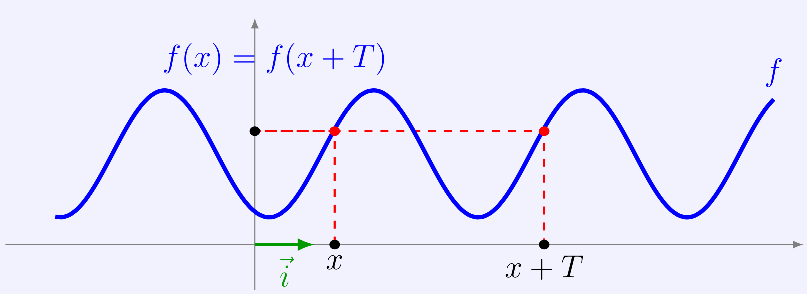
$$\forall x \in D_f : f(x + T) = f(x)$$

The smallest such T is called the fundamental period.

Definition 2.7

We say that f is a periodic function if there exists $k > 0$ where :

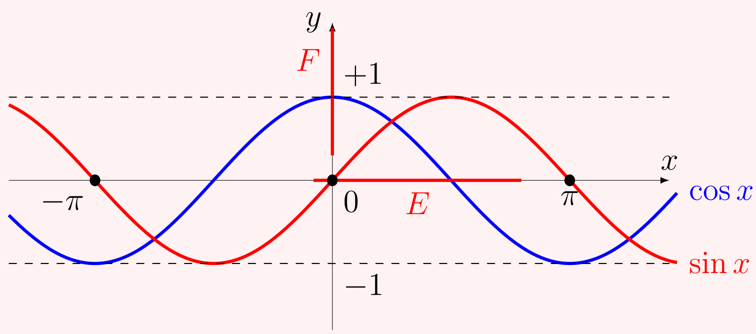
$$\forall x \in D_f : f(x + k) = f(x).$$



The part of the graph that corresponds to the smallest repeating pattern of a periodic function is called one cycle. The gap between two consecutive points that mark the end of the same cycle is called the period.

Example 2.10

The sine and cosine functions are periodic functions with a period of 2π, while the tangent function is a periodic function with a period of π.



2.2.3 Function Positivity and Negativity

Definition 2.8

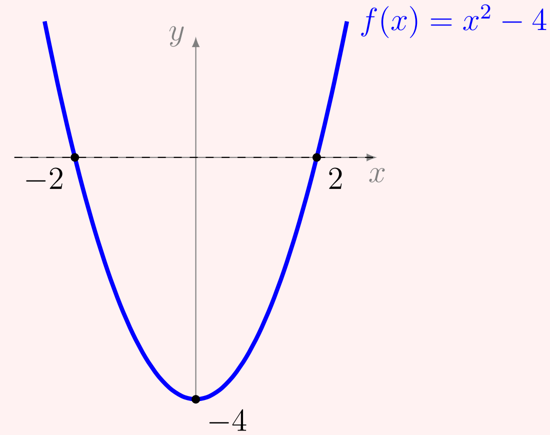
For $f : D_f \to \mathbb{R}$ and $\Delta \subseteq D_f$:

- f is **positive** on Δ if $\forall x \in \Delta : f(x) \geq 0$
- f is **strictly positive** on Δ if $\forall x \in \Delta : f(x) > 0$
- f is **negative** on Δ if $\forall x \in \Delta : f(x) \leq 0$
- f is **strictly negative** on Δ if $\forall x \in \Delta : f(x) < 0$

Example 2.11 (Sign Analysis)

Consider $f(x) = x^2 - 4$:

- Positive on $(-\infty, -2] \cup [2, \infty)$
- Negative on $(-2, 2)$
- Zero at $x = \pm 2$



Sign analysis of $f(x) = x^2 - 4$

Remark 2.2.1.

- *If the function f is positive, its graph lies above the x-axis, and conversely, if the function f is negative, its graph lies below the x-axis.*
- *If the function f is strictly positive or strictly negative, its graph never intersects the x-axis.*

2.2.4 Operations on functions

Let $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}$ be two defined functions on the same part U of the set $\mathbb{R}$. From this, we can define the following functions:

1) The sum of the functions f and g is the function $f + g : U \to \mathbb{R}$ defined as follows:

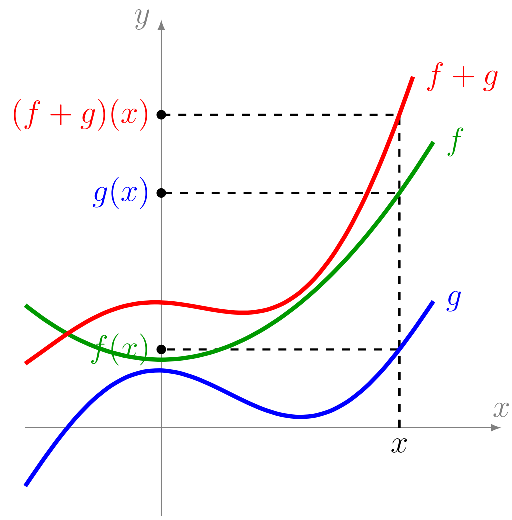
$$\forall x \in U, (f + g)(x) = f(x) + g(x).$$

2) The product of the functions f and g is the function $f \cdot g : U \to \mathbb{R}$ defined as follows:

$$\forall x \in U, (f \cdot g)(x) = f(x) \cdot g(x).$$

3) The product by scalar $\lambda \in \mathbb{R}$ and the function f is the function $\lambda \cdot f : U \to \mathbb{R}$ defined as follows:

$$\forall x \in U, (\lambda \cdot f)(x) = \lambda \cdot f(x).$$



2.2.5 Comparison of two functions

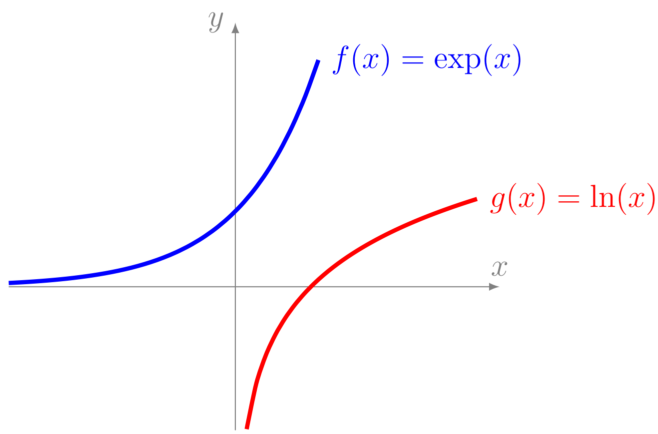
Let f and g be two defined functions on the same domain $\Delta \subset D_f \cap D_g$. We say that f is less than or equal to g, denoted as:

$$f \leq g: \quad \text{if} \quad \forall x \in \Delta, f(x) \leq g(x).$$

We say that f is greater than or equal to g, denoted as:

$$f \geq g: \quad \text{if} \quad \forall x \in \Delta, f(x) \geq g(x).$$

Remark 2.2.2. *If the function f is greater than or equal to g, then the graph of the function f lies above the graph of the function g.*



2.2.6 Function Monotony

The monotony of a function describes how a function behaves as its input increases: whether it consistently rises, falls, or remains constant. Understanding the monotony of functions is crucial in calculus, optimization, and many applied fields.

Let f be a function defined on its domain D_f, and let I be a subset of D_f.

Definition 2.9 (Increasing Function)

A function f is **increasing** on I if and only if:

$$\forall (x, y) \in I^2 : x > y \implies f(x) \geq f(y).$$

This means the function's output never decreases as the input increases. The function may have flat (constant) regions.

Example 2.12

The function $f(x) = \lfloor x \rfloor$ (floor function) is increasing on $\mathbb{R}$, though not strictly increasing. For example:

- At $x = 1.5$, $f(x) = 1$
- At $x = 2$, $f(x) = 2$
- At $x = 2.3$, $f(x) = 2$

We see that while the function never decreases, it remains constant between integer values.

Definition 2.10 (Strictly Increasing Function)

A function f is **strictly increasing** on I if and only if:

$$\forall (x, y) \in I^2 : x > y \implies f(x) > f(y).$$

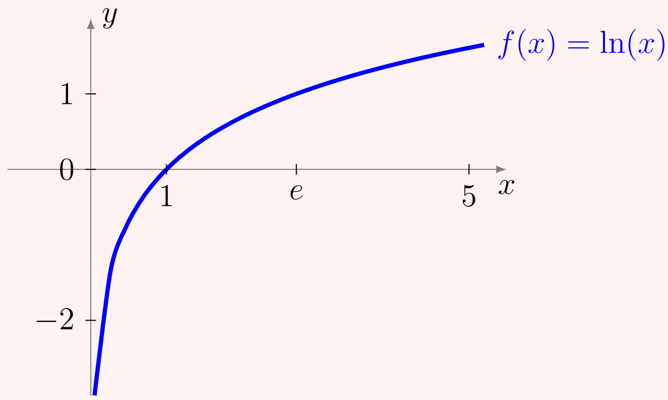
In this case, the function has no flat regions - it always increases as the input increases.

Example 2.13

The natural logarithm function $f(x) = \ln(x)$ is strictly increasing on its domain $]0, +\infty[$:

- $\ln(1) = 0$
- $\ln(e) \approx 1.718$
- $\ln(10) \approx 2.303$

As we can see, as x increases, $\ln(x)$ always increases.



Definition 2.11 (Decreasing Function)

A function f is **decreasing** on I if and only if:

$$\forall (x, y) \in I^2 : x > y \implies f(x) \leq f(y).$$

This means the function's output never increases as the input increases. The function may have flat regions.

Example 2.14

The function $f(x) = -|x|$ is decreasing on $\mathbb{R}$:

- $f(-2) = -2$
- $f(-1) = -1$
- $f(0) = 0$

- $f(1) = -1$
- $f(2) = -2$

Notice that between $x = -1$ and $x = 1$, the function increases and then decreases, but overall it's not monotonic on $\mathbb{R}$. However, it's decreasing on $[0, +\infty[$.

Definition 2.12 (Strictly Decreasing Function)

A function f is **strictly decreasing** on I if and only if:

$$\forall (x, y) \in I^2 : x > y \implies f(x) < f(y).$$

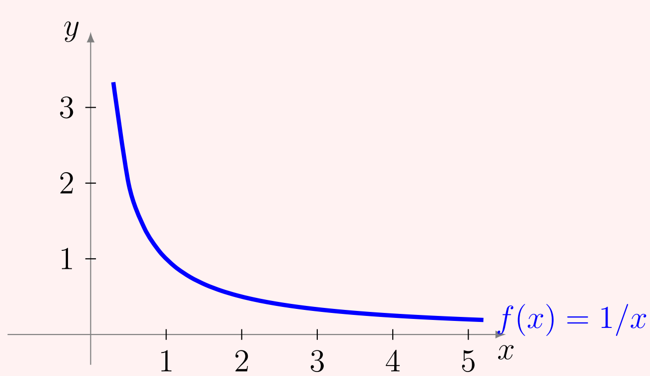
In this case, the function has no flat regions - it always decreases as the input increases.

Example 2.15

The reciprocal function $f(x) = \frac{1}{x}$ is strictly decreasing on $]0, +\infty[$:

- $f(1) = 1$
- $f(2) = 0.5$
- $f(10) = 0.1$

As x increases, $f(x)$ always decreases.



Remark 2.2.3. *It's important to specify the interval I when discussing monotony, as many functions have different behaviors on different intervals. For example:*

- *$f(x) = x^2$ is strictly decreasing on $]-\infty, 0]$ and strictly increasing on $[0, +\infty[$*

- *$f(x) = \sin(x)$ alternates between increasing and decreasing on different intervals*

Remark 2.2.4. *For differentiable functions, we can determine monotony using the derivative:*

- *If $f'(x) \geq 0$ on I, then f is increasing on I*

- *If $f'(x) > 0$ on I, then f is strictly increasing on I*

- *If $f'(x) \leq 0$ on I, then f is decreasing on I*

- *If $f'(x) < 0$ on I, then f is strictly decreasing on I*

2.2.7 Finite function

Before investigating whether a function is bounded or not, it must be defined on a non-empty set, and then we can start searching for the bounds of the function.

Definition 2.13

Let f be a numerical function defined on the set D_f

1) We say that f is bounded above if and only if there exists a real number M such that:

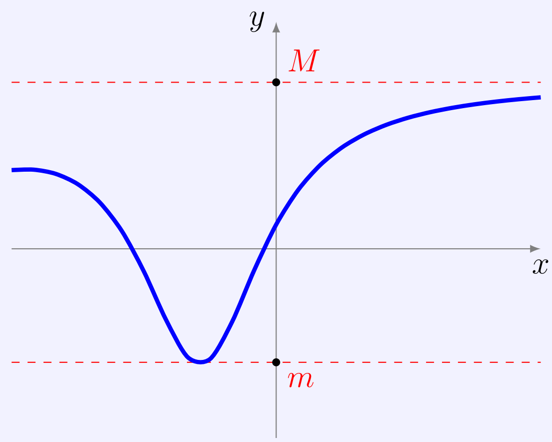
$$\forall x \in D_f : \quad f(x) \leq M.$$

2) We say that f is bounded below if and only if there exists a real number m such that:

$$\forall x \in D_f : \quad m \leq f(x).$$

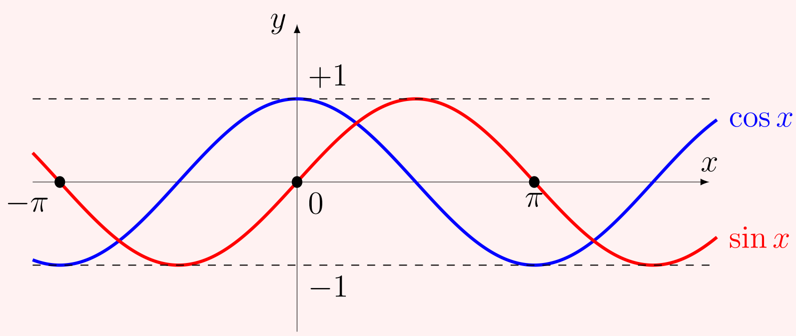
3) We say that f is bounded if and only if there exist two real numbers m and M such that:

$$\forall x \in D_f : \quad m \leq f(x) \leq M.$$



Example 2.16

The sine and cosine functions are bounded functions.



2.2.8 Max and min values of a function

Definition 2.14

Let f be a numerical function defined on the set D_f, and let $x_0 \in D_f$ and I be a subset of D_f.

1) We say that the number $f(x_0)$ is the absolute maximum value of the function f at the point x_0 if:

$$\forall x \in D_f : \quad f(x) \leq f(x_0).$$

2) We say that the number $f(x_0)$ is a relative maximum value of the function f at the

point x_0 in the domain I if $x_0 \in I$ and:

$$\forall x \in I \quad f(x) \leq f(x_0).$$

3) We say that the number $f(x_0)$ is the absolute minimum value of the function f at the point x_0 if:

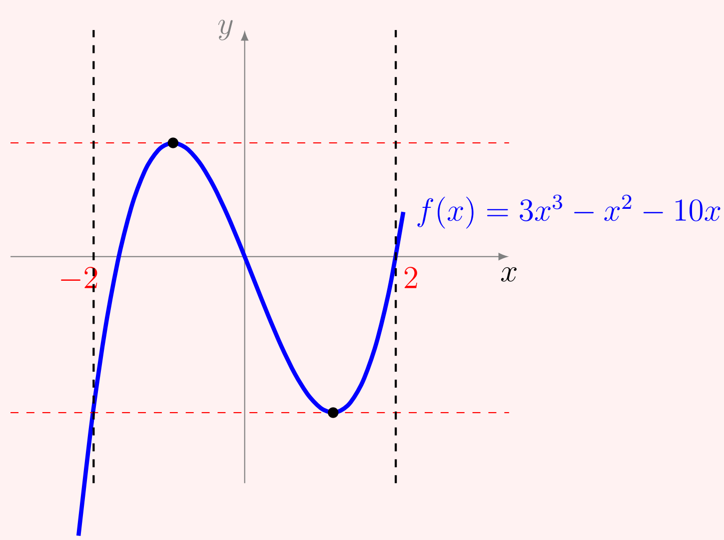
$$\forall x \in D_f \quad f(x) \geq f(x_0).$$

4) We say that the number $f(x_0)$ is a relative minimum value of the function f at the point x_0 in the domain I if $x_0 \in I$ and:

$$\forall x \in I \quad f(x) \geq f(x_0).$$

Example 2.17

The function f has an upper limit and a lower limit at the two specified points in the graph on the domain $[2, 2]$.



2.3 Limits

Limits are one of the most fundamental concepts in mathematical analysis, forming the foundation for calculus, continuity, differentiation, and integration. While you may have encountered limits in previous studies, this section provides a rigorous treatment with deeper insights and applications.

2.3.1 Definitions and Fundamental Concepts

Neighborhoods and Point Limits

Definition 2.15 (Neighborhood)

A subset $V \subseteq \mathbb{R}$ is called a **neighborhood** of a point $x_0 \in \mathbb{R}$ if there exists an open interval (a, b) containing x_0 such that $(a, b) \subseteq V$. In other words, V contains all points sufficiently close to x_0.

Example 2.18

- The interval $(1.9, 2.1)$ is a neighborhood of $x_0 = 2$
- The set $[0, \infty)$ is a neighborhood of $x_0 = 1$ but not of $x_0 = 0$
- The entire real line $\mathbb{R}$ is a neighborhood of every point

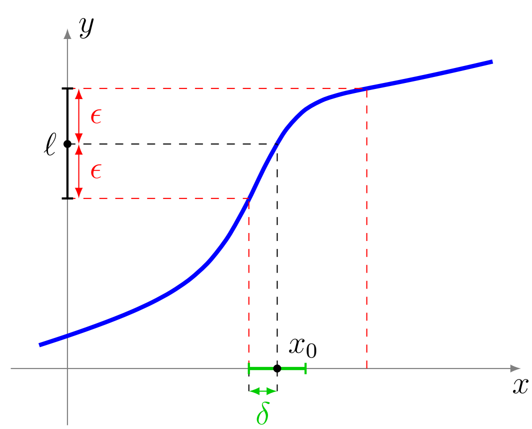
Definition 2.16 (Finite Limit at a Point)

Let $f : I \to \mathbb{R}$ be a function defined on some interval $I \subseteq \mathbb{R}$, and let $x_0 \in \mathbb{R}$ be a limit point of I. We say that f has **limit** $\ell \in \mathbb{R}$ as x approaches x_0 if:

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x \in I:\ 0 < |x - x_0| < \delta \implies |f(x) - \ell| < \epsilon.$$

We write this as:

$$\lim_{x \to x_0} f(x) = \ell \quad \text{or} \quad \lim_{x_0} f = \ell.$$



Example 2.19 (Linear Function)

Let $f(x) = 3x - 2$. We want to verify that $\lim_{x\to 1} f(x) = 1$ using the ϵ-δ definition.
Given any $\epsilon > 0$, we need to find $\delta > 0$ such that:

$$0 < |x - 1| < \delta \implies |(3x - 2) - 1| < \epsilon$$

Working backwards:

$$\begin{aligned} |3x - 3| &< \epsilon \\ 3|x - 1| &< \epsilon \\ |x - 1| &< \frac{\epsilon}{3} \end{aligned}$$

Thus, choosing $\delta = \frac{\epsilon}{3}$ satisfies the definition. For example:

- If $\epsilon = 0.3$, then $\delta = 0.1$
- If $\epsilon = 0.03$, then $\delta = 0.01$

This proves that $\lim_{x\to 1}(3x - 2) = 1$.

Example 2.20 (Quadratic Function)

Consider $f(x) = x^2$ at $x_0 = 2$. We claim $\lim_{x\to 2} x^2 = 4$.
For any $\epsilon > 0$, we need $|x^2 - 4| < \epsilon$ when $0 < |x - 2| < \delta$.
Notice that:

$$|x^2 - 4| = |x - 2||x + 2|$$

If we restrict $\delta \leq 1$, then $1 < x < 3$, so $|x + 2| < 5$. Thus:

$$|x^2 - 4| < 5|x - 2|$$

To make this $< \epsilon$, we need $|x - 2| < \epsilon/5$. Therefore, we take:

$$\delta = \min\left(1, \frac{\epsilon}{5}\right)$$

This shows how the choice of δ may depend on both ϵ and the behavior of the function near x_0.

Infinite Limits

Definition 2.17 (Infinite Limits at a Point)

Let f be defined on a punctured neighborhood of x_0.

- f tends to $+\infty$ as $x \to x_0$ if:

$$\forall M > 0,\ \exists \delta > 0,\ \forall x \in I:\ 0 < |x - x_0| < \delta \implies f(x) > M$$

  We write $\lim_{x\to x_0} f(x) = +\infty$.

- f tends to $-\infty$ as $x \to x_0$ if:

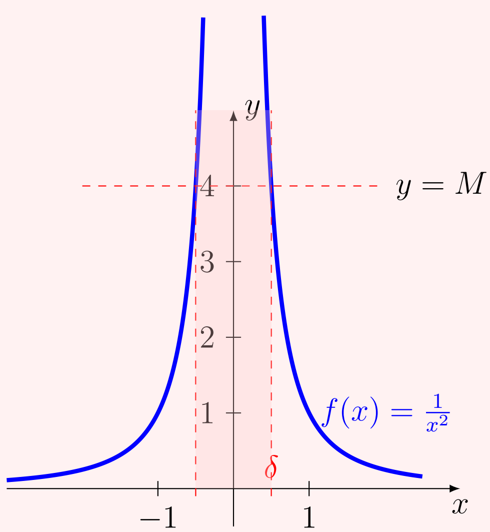
$$\forall M > 0,\ \exists \delta > 0,\ \forall x \in I:\ 0 < |x - x_0| < \delta \implies f(x) < -M$$

  We write $\lim_{x\to x_0} f(x) = -\infty$.

Example 2.21

The function $f(x) = \frac{1}{x^2}$ tends to $+\infty$ as $x \to 0$. For any $M > 0$, choose $\delta = \frac{1}{\sqrt{M}}$. Then:

$$0 < |x| < \delta \implies x^2 < \delta^2 = \frac{1}{M} \implies \frac{1}{x^2} > M$$



Limits at Infinity

Definition 2.18 (Finite Limits at Infinity)

Let f be defined on an interval $(a, +\infty)$.

- f converges to $\ell \in \mathbb{R}$ as $x \to +\infty$ if:

$$\forall \epsilon > 0,\ \exists B > 0,\ \forall x > B :\ |f(x) - \ell| < \epsilon$$

  We write $\lim_{x\to+\infty} f(x) = \ell$.

- Similarly, for f defined on $(-\infty, a)$, it converges to ℓ as $x \to -\infty$ if:

$$\forall \epsilon > 0,\ \exists B > 0,\ \forall x < B :\ |f(x) - \ell| < \epsilon$$

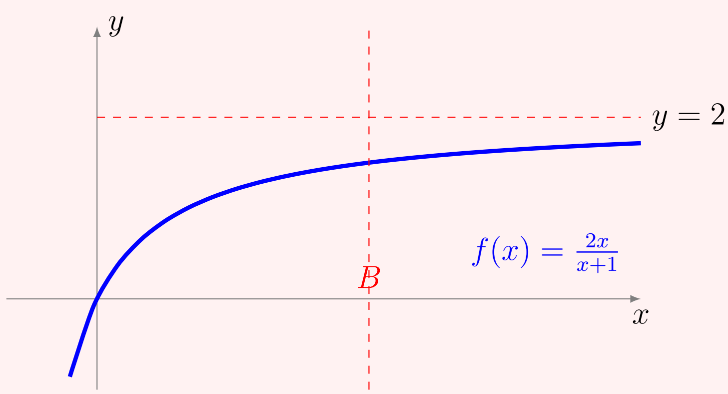
  We write $\lim_{x\to-\infty} f(x) = \ell$.

Example 2.22

The function $f(x) = \frac{2x}{x+1}$ approaches 2 as $x \to +\infty$.
For any $\epsilon > 0$, we need:

$$\left|\frac{2x}{x+1} - 2\right| = \left|\frac{-2}{x+1}\right| = \frac{2}{|x+1|} < \epsilon$$

This holds when $x > \frac{2}{\epsilon} - 1$. Thus, choosing $B = \max\left(0, \frac{2}{\epsilon} - 1\right)$ satisfies the definition.



Definition 2.19 (Infinite Limits at Infinity)

- f tends to $+\infty$ as $x \to +\infty$ if:

$$\forall M > 0,\ \exists B > 0,\ \forall x > B:\ f(x) > M$$

- f tends to $-\infty$ as $x \to +\infty$ if:

$$\forall M > 0,\ \exists B > 0,\ \forall x > B:\ f(x) < -M$$

Similar definitions apply for $x \to -\infty$.

Example 2.23

The function $f(x) = x^2$ tends to $+\infty$ as $x \to +\infty$. For any $M > 0$, choose $B = \sqrt{M}$. Then:

$$x > B \implies x^2 > B^2 = M$$

2.3.2 Operations on Limits and Limit Theorems

Theorem 2.1 (Limit Arithmetic)

Let f and g be functions with $\lim_{x\to x_0} f(x) = \ell$ and $\lim_{x\to x_0} g(x) = \ell'$ (where x_0 can be finite or $\pm\infty$). Then:

1. **Scalar Multiplication**: $\lim_{x\to x_0}[\lambda f(x)] = \lambda\ell$ for any $\lambda \in \mathbb{R}$
2. **Sum Rule**: $\lim_{x\to x_0}[f(x) + g(x)] = \ell + \ell'$
3. **Product Rule**: $\lim_{x\to x_0}[f(x) \cdot g(x)] = \ell \cdot \ell'$
4. **Quotient Rule**: If $\ell' \neq 0$, $\lim_{x\to x_0}\left[\frac{f(x)}{g(x)}\right] = \frac{\ell}{\ell'}$
5. **Reciprocal Rule**: If $\ell \neq 0$, $\lim_{x\to x_0}\frac{1}{f(x)} = \frac{1}{\ell}$

Example 2.24

Let $f(x) = x^2 + 3x$ and $g(x) = 2x - 1$. Then as $x \to 2$:

$$\lim_{x\to 2} f(x) = 4 + 6 = 10$$
$$\lim_{x\to 2} g(x) = 4 - 1 = 3$$
$$\lim_{x\to 2} \frac{f(x)}{g(x)} = \frac{10}{3} \quad \text{(by Quotient Rule)}$$

Theorem 2.2 (Squeeze Theorem)

If $f(x) \leq g(x) \leq h(x)$ for all x in a punctured neighborhood of x_0, and $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = L$, then $\lim_{x\to x_0} g(x) = L$.

Example 2.25

To find $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$, note that:

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

Since $\lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2 = 0$, by the Squeeze Theorem the limit is 0.

Theorem 2.3 (Continuity and Limits)

If f is continuous at x_0, then $\lim_{x\to x_0} f(x) = f(x_0)$. This applies to all elementary functions (polynomials, rational functions, trigonometric functions, exponential and logarithmic functions, etc.) at points in their domains.

Example 2.26

Since $f(x) = \sin(x)$ is continuous everywhere:

$$\lim_{x\to \pi/2} \sin(x) = \sin\left(\frac{\pi}{2}\right) = 1$$

2.4 Continuity

2.4.1 Continuity at a point

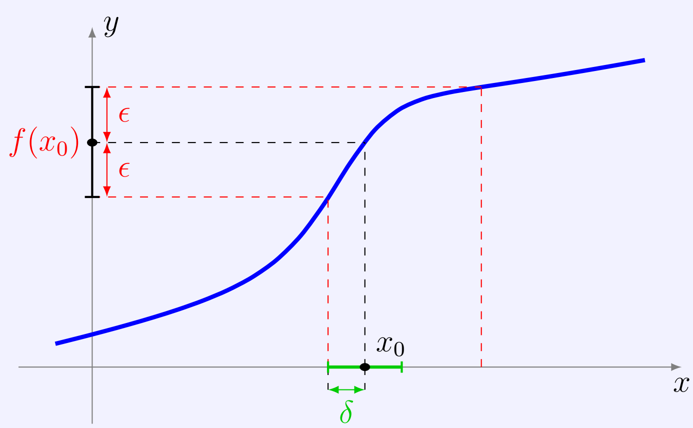
Definition 2.20

Let $f : I \to \mathbb{R}$ be a function defined on the domain I of the real numbers. Let $x_0 \in \mathbb{R}$ be a point in the domain I. We say that the function f is continuous at the point x_0 if the following holds:

$$\forall \epsilon > 0, \quad \exists \delta > 0, \quad \forall x \in I, \quad |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon,$$

we write:

$$\lim_{x \to x_0} f(x) = f(x_0).$$



Example 2.27

The function $f(x) = e^x$ is continuous at the point $x_0 = 0$ because

$$\lim_{x \to x_0} f(x) = \lim_{x \to 0} e^x = e^0 = 1 = f(x_0).$$

2.4.2 Continuity on domain

Definition 2.21

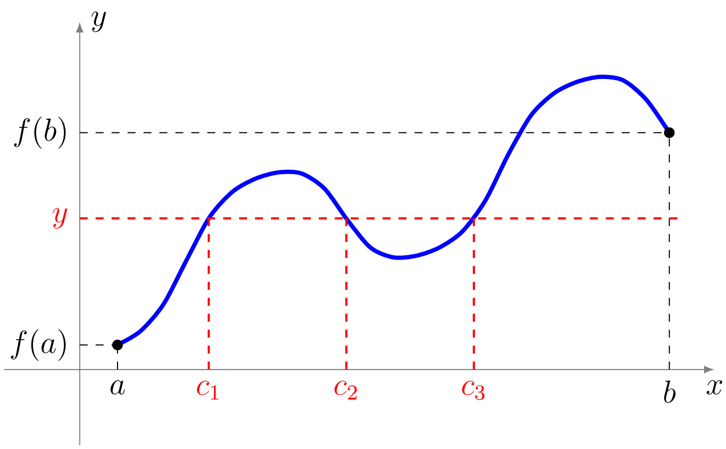
Let $f : I \to \mathbb{R}$ be a function defined on the domain I of $\mathbb{R}$.
We say that the function f is continuous on the domain I if it is continuous on all points of the domain I. We denote the set of continuous functions on the domain of I as $\mathscr{C}(I)$.

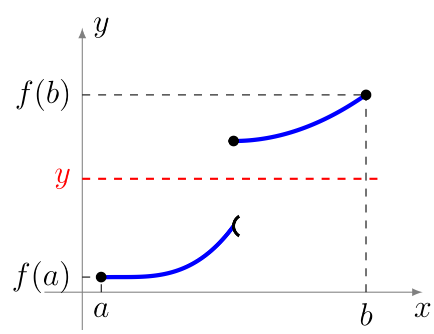
Mean Value Theorem

Theorem 2.4

Let $f : [a, b] \to \mathbb{R}$ be a function that is continuous on the closed interval $[a, b]$. For any real number y that lies between $f(a)$ and $f(b)$, there exists a real number $c \in [a, b]$ such that $f(c) = y$.

(In the left figure), the real number c is not necessarily unique. On the other hand, if the function is not continuous, then the theorem does not hold (as shown in the figure on the right).





2.4.3 Continuous extension

A continuous extension of a function allows us to extend its domain or range smoothly while preserving its continuity, enabling us to analyze its behavior in a broader context and overcome limitations imposed by its original definition.

Definition 2.22

Let the domain I, x_0 be the point from I and $f : I_{\{}x_0\} \to \mathbb{R}$ be a function.

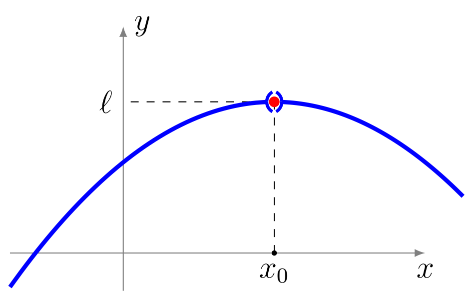
1) We say that the function f is continually extendable at the point x_0 if f accepts a finite limit at x_0, and we write:

$$\ell = \lim_{x_0} f.$$

2) We then define the function that we denote $\tilde{f} : I \to \mathbb{R}$ for each $x \in I$

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq x_0 \\ \ell & \text{if } x = x_0. \end{cases}$$

Then the function $\tilde{f}$ is continuous at point x_0, and the extension of the function f is called continuing at point x_0.



Example 2.28

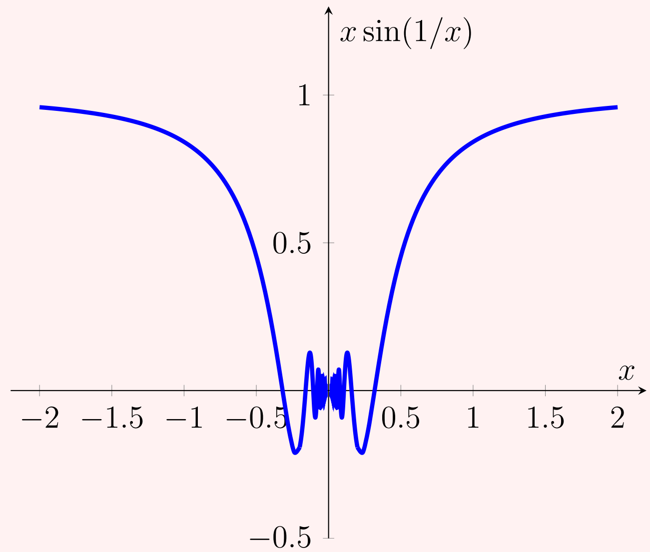
Let the function defined on the set $\mathbb{R}^*$ be as follows

$$f(x) = x \sin\left(\frac{1}{x}\right).$$

Does f accept extension by continuing at 0?

We have for each $x \in \mathbb{R}^*$ that $|f(x)| \leq |x|$, we get that f goes to 0 at 0. That is, it is extendable continuously at 0 and its extension is the function $\tilde{f}$ defined on $\mathbb{R}$ as follows:

$$\tilde{f}(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$



2.4.4 Operations on continuous functions

The primary operations on continuity are immediate consequences of analogous issues at the endpoints.

Proposition 2.4.1. *Let the two functions $f, g : I \to \mathbb{R}$ be given. Let $x_0 \in I$ be a point, hence:*

- *$\lambda \cdot f$ is continuous at x_0 ($\forall \lambda \in \mathbb{R}$).*
- *$f + g$ is continuous at x_0.*
- *$f \cdot g$ is continuous at x_0.*
- *If $f(x_0) \neq 0$, then $\frac{1}{f}$ is continuous at x_0.*

Proposition 2.4.2. *Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be two functions, where $f(I) \subset J$. If f is continuous at the point $x_0 \in I$ and g is continuous at the point $f(x_0)$, then the composite function $g \circ f$ is continuous at the point x_0.*